Approximate solution to one–dimensional phase change problem with non–uniform initial temperature via homotopy perturbation approach by

Radhi Ali Zaboon¹, Ahmed Ismail Mohammed²

¹Department of Mathematics, College of Science, AL-Mustansiriya University, Baghdad, Iraq
²Department of Mathematics, College of Basic Education, Misan University, Misan, Iraq.

ABSTRACT

In this article, Homotopy Perturbation Method is modified to some moving boundary value problem in order to obtain an approximate explicit solution with regard to phase change problem with a non-uniform initial temperature distribution. The initial approximations of homotopy perturbation method have chosen so that the initial and boundary conditions of the moving boundary value problem are satisfied. The obtained results are tabulated and graphically compared with those due to some authors. The numerical simulation using present approach show very good agreements with others, even with very small number of bases have being adapted.

Keywords: Homotopy perturbation method, moving boundary value problem of partial differential equation.

1. INTRODUCTION

The term moving boundary problems (MBP’s) is commonly used when the boundary is associated with time dependent problems and the boundary of the domain is not known in advance but has to be determined as a function of time and space. Moving boundary problem have received much attention due to their practical importance in engineering and science[19].These problems become nonlinear due present of moving boundary [8] and for this reason their analytical explicit solution are difficult to obtain in general.

Stefan problems (phase change problems) is one class of moving boundary value problem and as well as application, See Crank[9] and Hill[12]. The class of Stefan problem (MBP’S) is interesting because of its nonlinearity nature that is associated with the moving interface (see [8]). Due to presence of moving interface their exact solution are limited. Therefore, Many approximate solutions have been used to solve this problem numerical [5],[6],[24-27]. Stefan problems with time-dependent boundary condition require some special techniques one can see [27], [28], [31]. Savovic and Caldwell [30] presented finite difference solution of one-dimensional Stefan problem with periodic boundary conditions. Ahmed [4] discussed a new algorithm for moving boundary problem subject to periodic boundary conditions. In 2009, Rajeev et al. [23] used variational iteration method to solve a phase change problem with time dependent boundary condition and the result is obtained in term of Mittag-Leffler function. In 2012 Rajeev and M.S. Kushwaha [22] used adomaian decomposition method to solve a Stefan problem with periodic boundary condition.

In this paper an approximate explicit approach is interested via a Homotopy perturbation method with some modifications. The obtained results are compared with the non-classical variational solution obtained in [7]. Since both methods are based on the selection a suitable bases that approximate the solution.

2. DESCRIPTION OF THE PROBLEM (SOLIDIFICATION OF WATER)

The problem concerns heat transfer in an ice-water medium occupying the region \( 0 \leq x \leq 1 \). At any time \( t \). The water that undergoing phase change, is contained in the region \( (z_1(0) \leq x \leq z_2(0)) \) and the rest of the region outside it, is occupied by ice. Initially, \( z_1(0) = 0.23 \), \( z_2(0) = 0.73 \), and the temperature of ice is linear in each of two region in which it lies. The temperature of the water is assumed to be equal to zero which is also the critical temperature of phase change. The fixed surface \( x=0 \) and \( x=1 \), are maintained at unit negative temperature throughout [7].

Remarks 1:
1. As mentioned in [21], [29] due to the symmetry about of the problem \( x = 0.5 \), the problem is reduced to find its solution is the region \( 0 \leq x \leq 0.5 \), on the initial condition.
2. The region of initial condition is then written as non-uniform initial temperature region as following:
   \( 0 \leq x \leq 0.25 \), and \( 0.25 \leq x \leq 0.50 \)
3. The initial condition may be obtained as follows set \( u(x,0) = u_0 + (x-0.25) \) so that the
3.1. \( u(0,0) = -1 \)
3.2. \( u(x,0) = \frac{4x - 1}{0, \quad 0 \leq x \leq 0.25} \)

4. The water that undergoing phase change, is then contained in the region
\( 0 < x < s(t) \)

5. Using the non-dimensional form (see[11],[21],[29]) the following moving boundary problem value of Stefan type with non-uniform initial temperature is formulation as follows:

\[
\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0
\]  

Subject to the boundary conditions:
\[
u(0,t) = -1, \quad t > 0
\]
\[
u(s(t),t) = 0, \quad t > 0
\]
\[
\frac{\partial u(s(t),t)}{\partial x} = \frac{ds(t)}{dt}, \quad t > 0
\]

With non-uniform initial conditions
\[
u(x,0) = \begin{cases} \frac{4x - 1}{0, \quad 0 \leq x \leq 0.25} \end{cases}
\]

And the moving boundary is subject to:
\[
s(0) = 0.25
\]

Where \( u(x,t) \) is the temperature at distance \( x \) and time \( t \), \( s(t) \) being the position of the interface at time \( t \)

3. ANALYSIS OF HOMOTOPY PERTURBATION METHOD [13],[17],[20]

Let \( X \) and \( Y \) be the topological spaces. If \( f \) and \( g \) are continuous maps of the space \( X \) into \( Y \), it is said that \( f \) is homotopic to \( g \), if there is continuous map \( \mathcal{F} : X \times [0,1] \rightarrow Y \) such that \( \mathcal{F}(x,0) = f(x) \) and \( \mathcal{F}(x,1) = g(x) \), for each \( x \in X \), then the maps is called homotopy between \( f \) and \( g \). The homotopy perturbation method is a combination of classical perturbation technique and the homotopy map used in topology. To explain the basic idea of the homotopy perturbation method for solving non-linear differential equations, the following are considered

\[
A(u) - f(r) = 0, \quad r \in \Omega
\]

Subject to boundary condition
\[
B \left( \frac{\partial u}{\partial n} \right) = 0, \quad r \in \Gamma
\]

Where, \( A \) is a general differential operator, \( B \) is a boundary operator, \( u \) is a known analytical function, and \( \Gamma \) is the boundary of the domain \( \Omega \) (the computational domain). The operator \( A \) can be divided into two parts \( L \) and \( N \), where \( L \) is linear, while \( N \) is nonlinear. Therefore (7) can be rewritten as follows:

\[
L(u) + N(u) - f(r) = 0
\]

By the homotopy technique, the following are obtained construct a homotopy defined as

\[
v(r, p) : \Omega \times [0,1] \rightarrow R
\]

which satisfies:

\[
H(v, p) = (1 - \mu)pL(v) - L(u_0) + p[La(v) - f(r)] = 0
\]

Or

\[
H(v, p) = L(v) - L(u_0) + p[La(v) - f(r)] = 0
\]

Where, \( \mu \in \Gamma \) and \( p \in [0,1] \) is an embedding parameter, \( u_0 \) is an initial approximation of equation (7), which satisfies the boundary conditions. By (10) or (11), it easily follows that:

\[
H(v,0) = L(v) - L(u_0) = 0
\]

and the changing process of \( p \) from zero to unity is just that of \( H(v,p) \) from \( L(v) - L(u_0) \) to \( A(v) - f(r) \). In topology, this is called deformation, \( L(v) - L(u_0) \) and \( A(v) - f(r) \) are called homotopy. The embedding parameter \( p \) is introduced much more naturally, unaffected by artificial factors. Furthermore, it can be considered as a small parameter for \( 0 < p \leq 1 \). By applying the perturbation technique one assume that the solution of (10) and (11) can be expressed as a power series of \( p \), [3], [18], i.e:

\[
v = v_0 + pv_1 + p^2v_2 + \cdots
\]

Therefore, the approximate solution of (7) can be readily obtained as follows:

\[
u = u_0 - pL(u_0) + p^2L(u_0) + \cdots
\]

The series of equation (15) is convergent for most of the cases. However, the convergent rate depends on the nonlinear operator \( N(p) \). the following suggestions have already been made where details can be shown in [16].
1. The second derivative of $N(x)$ with respect to $u$ must be small because the parameter may be relative large i.e $\varphi \rightarrow 1$

2. The norm of: $L^{-1}\left(\frac{F(u)}{u}\right)$ must be smaller than one so that the series is convergent.

4. ADOMIAN POLYNOMIALS (SIMPLE ALGORITHM) [14]
The Adomian decomposition method is a technique for solving functional equations in the form:

$$u = g + F(u)$$  \hspace{1cm} (16)

In some functional space, say $G$. The solution $u$ is considered as the summation of a series, say;

$$u = \sum_{n=0}^{\infty} u_n$$  \hspace{1cm} (17)

And $F(u)$ as the summation of a series, say;

$$F(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n)$$  \hspace{1cm} (18)

Where $A_n$’s, called Adomian polynomials, has been introduced by the Adomian himself by the formula:

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!} \frac{d^n}{dx^n} \left[ F\left( \sum_{m=0}^{n} u_m A_m \right) \right]_{x=0}$$  \hspace{1cm} (19)

Many computational algorithms are available to compute adomain polynomial, for example one can see [2],[10],[15],[32]. In [14] given a suitable and simpler one, so, we have adopted this algorithm and as follows for calculating $A_0, A_1, \ldots, A_n$

\begin{itemize}
  \item [step1]: Input nonlinear term $F(u)$ and $n$, the number of Adomian polynomials needed.
  \item [step2]: Set $A_0 = F(u_0)$
  \item [step3]: For $n = 0$ to $n = 1$ do:
    \begin{enumerate}
      \item $A_0(u_0, u_1, \ldots, u_n) = A_0(u_0) u_{n+1} + u_{n+1} + (n+1) u_{n+3} A_{n+3}$
      \item $A_n : \forall t = 0, 1, \ldots, k$
    \end{enumerate}
  \item [step4]: Taking the first order derivative of $A_n$, with respect to $\lambda$, and then let $\lambda = 0$
  \item [step5]: Output $A_0, A_1, \ldots, A_n$.
\end{itemize}

According to the above Algorithm, Adomian polynomials will be computed as follows:

$$A_0 = F(u_0)$$

$$A_1 = \frac{d}{dx} F(u_0 + u_1) \bigg|_{x=0} = u_1 F'(u_0),$$

$$A_2 = \frac{d}{dx} \left( \frac{d}{dx} F(u_0 + 2u_1 + u_2) \bigg|_{x=0} \right) = u_2 F'(u_0) + \frac{u_1^2}{2!} F''(u_0).$$

And so on. The components of $u_n$, $n \geq 1$.

**Remarks 2:**

Based on the problem formulation (1-6) and our choice of the linear operator $L$ of the (HPM), as discussed in (9) the following options are firstly discussed

1. If one choose $L$ as the whose linear operators of the problem (1-6) as

$\begin{align*}
L & \equiv \frac{\partial}{\partial x} - L_{1g}, \\
\frac{\partial}{\partial x} & \equiv 0.
\end{align*}$

The inverse operator $L^{-1}$ is difficult to obtain, hence this option is omitted

2. If one can choose $L \equiv L_2$, and $u(x) = e^{\frac{x}{2\sigma}}$

The trivial solution is obtained.

3. On setting $L$ of (9) as $L \equiv L_{1g}$ and including the remainder operator $L_2$ in the nonlinear part $N(u)$ of (HPM), (9), this option is adapted and as follows:

5. SOLUTION OF THE PROBLEM (1-6) VIA (HPM)

Based on remarks (2), the following is assumed

$$L_{1g} u + \frac{\partial}{\partial x} u - L_{1g} u(x, t), \hspace{0.5cm} 0 < x < s_0(t), \hspace{0.5cm} t > 0$$  \hspace{1cm} (21)

Where $L_{1g} = \frac{\partial^2}{\partial x^2}, \hspace{0.5cm} L_2 = \frac{\partial}{\partial t}$.

Since two condition on $x \rightarrow 0$ and $x \rightarrow s_0(t)$ are assumed, then the following expression is suggested
\[ u_0(x, t) = ax + b \]

On setting \( u_0(0, t) = -1 \), \( u_0(t, t) = 0 \)

From these equation, we have got
\[ a = \frac{1}{s_0(t)} \text{ and } b = -1 \]

Thus \( u_0(x, t) = \left( \frac{x}{s_0(t)} - 1 \right) \) has selected

\( s_0(t) \) is also chosen so that \( s(0) = 0.25 \), one can choose the following
\( s_0(t) = 0.25 + 2t \)

From (10), we have got
\[ H(w, p) \equiv (1 - p) \left[ \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} \right] + p \left[ \frac{\partial^2 v}{\partial x^2} - \frac{\partial w}{\partial t} \right] = 0 \tag{22} \]

\[ v = v_0 + p^2 v_1 + p^3 v_2 + p^4 v_3 + \ldots \tag{23} \]

Setting \( p = 1 \) in (23), one gets
\[ u = \lim_{p \to 1} v, \quad v = v_0 + v_1 + v_2 + v_3 + \ldots \tag{24} \]

\[ \frac{\partial^2 v}{\partial x^2} - (1 - p) \frac{\partial^2 u_0}{\partial x^2} + p \left( \frac{\partial v}{\partial t} \right) = u \tag{25} \]

From \( u_0 = \left( \frac{x}{s_0(t)} - 1 \right) \) and equating the coefficients of \( p \) to zero, the following are obtained:

Coefficient of \( p^0 \) gives
\[ \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0 \tag{26} \]

It is clear that from (26),
\[ v_0(x, t) = u_0(x, t) = \left( \frac{x}{s_0(t)} - 1 \right) \tag{27} \]

Coefficient of \( p^2 \):
\[ \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial v_2}{\partial x} = 0 \]

\[ \frac{\partial^2 v_2}{\partial x^2} = -\frac{\partial^2 u_0}{\partial x^2} \]

\[ v_2(x, t) = \int \int \frac{\partial^2 u_0}{\partial x^2} dx dx = \int \int \left( \frac{2x}{(2t + 0.25)^2} \right) dx dx \]

\[ v_2(x, t) = \frac{-16x^3}{3(8t + 1)^2} \tag{28} \]

Coefficient of \( p^3 \):
\[ \frac{\partial^2 v_3}{\partial x^2} - \frac{\partial v_3}{\partial t} = 0 \text{ from (28) yields: } \quad v_3(x, t) = \frac{64x^5}{15(8t + 1)^3} \tag{29} \]

Coefficient of \( p^4 \):
\[ \frac{\partial^2 v_4}{\partial x^2} - \frac{\partial v_4}{\partial t} = 0 \text{ from (28) yields: } \quad v_4(x, t) = \frac{-256x^7}{105(8t + 1)^4} \tag{30} \]

Coefficient of \( p^5 \):
\[ \frac{\partial^2 v_5}{\partial x^2} - \frac{\partial v_5}{\partial t} = u \text{ from (28) yields: } \quad v_5(x, t) = \frac{\partial v_5(x, t)}{\partial t} \tag{31} \]

On setting
\[ v(x, t) = v_0(x, t) + p^2 v_1(x, t) + p^3 v_2(x, t) + p^4 v_3(x, t) + \ldots \tag{32} \]

\[ u(x, t) = \lim_{p \to 1} v(x, t), \quad \text{see (15)} \tag{33} \]

From the completed above, we have got
\[ u(x, t) = \left( \frac{x}{s_0(t)} - 1 \right) + \left( \frac{-16x^3}{3(8t + 1)^2} \right) + \left( \frac{64x^5}{15(8t + 1)^3} \right) + \left( \frac{-256x^7}{105(8t + 1)^4} \right) + \ldots \tag{34} \]

From (4), the Stefan condition for this problem is very interesting in testing the results then on setting:
\[ \frac{\partial u(x, t)}{\partial t} = \frac{ds_0(t)}{\partial t} \]

On integration both side with respect to \( t \) from 0 to \( t \) one get
\[ \int_{0}^{1} \frac{dx}{\sqrt{x}} = \int_{0}^{1} \frac{dx}{\sqrt{1-x^2}} \]

\[ s(t) = s(0) + \int_{0}^{t} \frac{ds(u)}{dt} \, du \]

When the initial approximation is assumed\( s(t) = 0.25 \, t \leq t \leq 1 \)

From (35) and (36), we have got

\[ \sum_{i=0}^{n} s_{i}(t) = s(t) + \left( \sum_{i=0}^{n} A_{i} \right) dx \]

And so on, the components of \( s(t) \), \( n \geq 1 \), can be completely determined as follows:

\[ s_{n-1} = \int_{0}^{1} A_{n-1} \, dt \]

And so on. The approximate explicit solution of the moving \( s(t) \) of the problem (1)-(6) is then obtained by:

\[ s(t) = s_{0} + s_{1} + \ldots \]

To define an accuracy criterion of this approach, the Stefan condition (4) is used and as follows:

\[ \frac{\partial s_{i}(x,t)}{\partial t} = \frac{d s_{i}(x,t)}{d x} \]

And

\[ \frac{10}{240} + \frac{3 \ln 2}{2} + \frac{3 t^{2}}{10} - \frac{4 t^{3}}{45} + \ldots \]
\[
\frac{ds(t)}{dt} = \frac{1}{2} \delta^2 - \frac{1}{2} \frac{d^2}{dx^2} \frac{d^2}{dt^2} - \frac{2}{3} \frac{d^2}{dx^2} \frac{d}{dt} - \frac{3}{2} \frac{d^2}{dx^2} \frac{1}{2} \frac{d}{dt} \frac{d^2}{dt^2} + \frac{211}{240} \frac{d^2}{dx^2} \frac{d}{dt}.
\]

(41)

From (40) and (41), we define the following error criterion and it is called an absolute error for stefan condition, i.e.

\[
\text{Absolute error} = \left| \frac{\partial u(x,t)}{\partial x} - \frac{\partial s(t)}{\partial x} \right|.
\]

(42)

We have used the absolute error (42) and we adjust the number of bases for \(u(x,t)\) and \(s(t)\) as follows:

\[
u(x,t) = \sum_{i=1}^{n_v} u_i(x,t)
\]

(43)

\[
s(t) = \sum_{i=1}^{n_s} s_i(t)
\]

(44)

Based on the following simulation, the number \(n_v\) and \(n_s\) are selected.

The simulation of descritized time-interval \(t \in [0.0, 1.0]\), is shown below.

<table>
<thead>
<tr>
<th>t</th>
<th>(\partial u(x,t))</th>
<th>(\frac{\partial s(t)}{\partial x})</th>
<th>Absolute error</th>
</tr>
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<td>3.079</td>
<td>0.0</td>
</tr>
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</tr>
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<td>1.35872222</td>
<td>0.01572222</td>
</tr>
</tbody>
</table>

6. NUMERICAL COMPARISONS

On selection the number \(n_v\) and \(n_s\), we have checked the error for \(n_v = 1, n_s = 1, 2, 3\), and then \(n_s = 3\) have a reasonable absolute error as show below. Thus \(n_v = 1, n_s = 3\) have been adapted for simplicity. One can also increase the accuracy by selecting more bases in \(u(x,t)\) and \(s(t)\), i.e. \(n_v > 1, n_s > 3\).

\[
u(x,t) = \left( x + \frac{1}{s_2(s_1-x)} \right) + \left( \frac{16x^2}{3(s_1+1)^3} \right) + \left( \frac{15(s_1+1)^3}{105(s_1+1)^3} \right) + \left( \frac{15(s_1+1)^3}{105(s_1+1)^3} \right) + \cdots
\]

\[
u(t) = 0.25 + \left( \frac{1}{2} \frac{d^2}{dt^2} - \frac{2}{3} \frac{d^2}{dx^2} \frac{d}{dt} \right) + \left( \frac{211}{240} \frac{d^2}{dx^2} \frac{d}{dt} \right) + \cdots
\]

The comparison have been implemented with the non-classical variational solution of a one dimensional phase change problem with non-uniform initial temperature, submitted by Radhi A. Zaboon in [7], pp.471-479. The approximate solution of problem (1)-(6) is defined and obtains as:

\[
\Delta(x,t) = w(x,t) + \Delta(x) - s(t) \sum_{i=1}^{n_v} q_i G i(x,t), \quad n_v = 5
\]

\[
s(t) = 0.25 + \sum_{i=1}^{n_v} b_i s_i(t), \quad n_v = 4.
\]

Where the coefficient \(a_i, b_i\) are computed as:

\[
\begin{array}{c|c|c}
\hline
a_i & b_i \\
\hline
-97.897200000013 & -315.2350000000003 \\
164.84590000009 & 113.011000000047 \\
-207.870400000011 & -196.781200000011 \\
446.79659998542 & 136.728900000084 \\
\hline
\end{array}
\]
Thus
\[ f(x) = 0.25 + 4t - 315.22500000000003t^2 + 113.0112000000047t^3 - 295.78122300011t^4 + \\
136.726900000006t^5 + \frac{dx}{dt}(t) - 1 + \left(1 - \frac{dx}{dt}(t)\right) + \left(\frac{dx}{dt}(t)\right)^2 + t(t - s(t))^2 - 97.807000000013 + 146.4458000000098t - \\
207.870600000011x + 440.76599998542t^2t - \\
406.89499999000000t^3 \] \tag{45}

From (34) and (39), (45) and (46), the following comparisons are made, where \( u(x, t) \) and \( s(x) \) are computed using the present approach.

Table (2)

| t    | Non-classical v. m. \( n_x = 5, n_t = 4 \) | The present method \( n_x = 3, n_t = 1 \) | Absolute of error \( |f(x) - s(x)| \) |
|------|-------------------------------------------|-----------------------------------------------|------------------|
| 0    | 0.25                                      | 0.25                                          | 0.0              |
| 0.0172 | 0.31003                                   | 0.299                                         | 0.01103          |
| 0.0379 | 0.36207                                   | 0.348                                         | 0.01407          |
| 0.0619 | 0.40085                                   | 0.395                                         | 0.00585          |
| 0.0892 | 0.42450                                   | 0.439                                         | 0.0145           |

The numerical results for different value of \( x \) and \( t \), with comparison

Table (3), the numerical results for different value of \( x \) and \( t \), with comparison

| \( T \) | \( x \) | \( s(x, t) \) by non-classical variational method \( n_x = 5, n_t = 4 \) | \( u(x, t) \) by present method \( n_x = 3, n_t = 1 \) | Absolute of error \( |u(x, t) - s(x, t)| \) |
|-------|--------|---------------------------------------------------------------|-----------------------------------------------------------------|------------------|
| 0     | 0      | \(-1\)                                                        | \(-1\)                                                          | 0.0              |
| 0.05  | -0.8   | -0.80066533                                                   | -0.80066533                                                     | 0.00066533       |
| 0.10  | -0.6   | -0.60529091                                                   | -0.60529091                                                     | 0.00529091       |
| 0.15  | -0.4   | -0.41768016                                                   | -0.41768016                                                     | 0.01768016       |
| 0.20  | -0.2   | -0.24133254                                                   | -0.24133254                                                     | 0.04133254       |
| 0.25  | 0.0    | 0.07931547                                                    | 0.07931547                                                      | 0.07931547       |
| 0.0172| 0      | \(-1\)                                                        | \(-1\)                                                          | 0.0              |
| 0.05  | -0.83593594 | -0.82470552                                        | -0.82470552                                                     | 0.01123042       |
| 0.10  | -0.67202132 | -0.65247488                        | -0.65247488                                                      | 0.01954644        |
Remarks 3:
1. From table (3) one can see the accuracy is very good even a small number of basis for (HPM), when \( n_1 = 3, n_2 = 1 \), is suggested to increase the accuracy, one can suggest the increasing of the numbers \( n_1 \) and \( n_2 \).
2. Since the boundary conditions of the problem in both methods are selected in the basis form (the approximate solution satisfying the boundary condition exactly). This numerical absolute error is zero, as show in table (3).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( a_{11} )</th>
<th>( a_{12} )</th>
<th>( a_{13} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>-0.50851096</td>
<td>-0.48626516</td>
<td>0.0222458</td>
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<tr>
<td>0.20</td>
<td>-0.34605014</td>
<td>-0.32882563</td>
<td>0.01722451</td>
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<tr>
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<td>-0.18260817</td>
<td>0.00332885</td>
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<td>0.0379</td>
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<td>-1.0</td>
<td>0.0</td>
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<td>-0.85034089</td>
<td>-0.84692355</td>
<td>0.00341734</td>
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<td>0.10</td>
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<td>-0.69618437</td>
<td>0.00641115</td>
</tr>
<tr>
<td>0.15</td>
<td>-0.55679114</td>
<td>-0.55004854</td>
<td>0.00674262</td>
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<tr>
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<tr>
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<td>-0.27452755</td>
<td>0.00536732</td>
</tr>
<tr>
<td>0.0619</td>
<td>0.0</td>
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<td>0.0</td>
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<td>0.15</td>
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<tr>
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<tr>
<td>0.25</td>
<td>-0.31206187</td>
<td>-0.36725164</td>
<td>0.05518977</td>
</tr>
</tbody>
</table>

Remark 4:
Fingers 1, 2, 3 and 4 presents the numerical comparison of \( a_1(x, t) \) and \( a_2(x, t) \) for different value of time \( t \).

7. CONCLUSIONS
The Homotopy perturbation method and decomposition method are successfully applied to find an approximate explicit expressions of temperature distribution in liquid region and interface position of a Stefan problem(1-6), respectively, the
initial approximations of \( u(0)=0 \) and \( u_{x}(0)=0 \) and a function to achieve both initial and boundary condition of the original problem (1-6) and that helps us to achieve results compared with the methods mentioned. Even a small number of bases are selected \( n_{b}=3, m_{b}=1 \).

REFERENCES


