On Characterization of b-open sets in a topological spaces.

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Abstract:
The purpose of this paper is to establish & project the theorems which exhibit the characterization of b-open sets in topological spaces and obtain some of interesting properties of b-open sets. We establish the interrelationship between b-closure & b-interior along with their representations in terms of scl, pcl, sint & pint. Also the class of b-open sets with regard to T &Tb has been worked out to be same.

Keywords: pre-open, semi-open, semi-pre-open, α-open, b-open, b – interior, p, s,α & sp-closure, p, s, α & sp-interior.

§1.Introduction
The mathematical papers [1]&[2] introduce and investigate semi-pre-open sets and b-open sets which are some of the weak forms of open sets and the complements of these sets are obviously the same type of closed sets.

In recent years a number of generalizations of open sets have been considered in the literature. Three of these notions were defined similarly using the closure operator (cl) and the interior operator (int) in the following manner & which are useful in the sequel:

Definition (1.1):
A subset A of a topological space (X,T) is called a pre-open set if

\[ A \subseteq \text{int}(\text{cl}(A)) \] and pre-closed if \( \text{cl}(\text{int}(A)) \subseteq A \).

Definition (1.2):
A subset A of a topological space (X,T) is called a semi-open set if

\[ A \subseteq \text{cl}(\text{int}(A)) \] and semi-closed if \( \text{int}(\text{cl}(A)) \subseteq A \).

Definition (1.3):
A subset A of a topological space (X,T) is called an α-open set if

\[ A \subseteq \text{int}(\text{cl}(\text{int}(A))) \] and an α-closed set if \( \text{cl}(\text{int}(\text{cl}(A))) \subseteq A \).

The first three notions are due to (a)[4],(b)[5] & (c)[6] respectively.

The concept of a pre-open set was introduced by H.H.Corson & E.Michael in the paper [7], where the term was used as “locally dense”.

The semi-pre-open set, called by D.Andrijevic, was conceptualized under the name “β-open”[8] by M.E .Abd El-Monsif etc.

Definition (1.4):
A subset A of a topological space (X,T) is called a semi-pre-open set (β-open set) if \( A \subseteq \text{cl}(\text{int}(\text{cl}(A))) \) and semi-pre-closed (β-closed)

if \( \text{int}(\text{cl}(\text{int}(A))) \subseteq A \).

§ b-open & b-closed sets:
Now, we consider a new class of generalized open sets given by D.Andrijevic under the name “b-open sets ” as:

Definition (1.5):
A subset A of a topological space (X,T) is called a b-open set if

\[ A \subseteq \text{cl}(\text{int}(A)) \bigcup \text{int}(\text{cl}(A)) \] and a b-closed if \( \text{cl}(\text{int}(A)) \bigcap \text{int}(\text{cl}(A)) \subseteq A \).

All the above given definitions are different and independent as illustrated by the following example:

Example (1.1):
Consider the set R of real numbers with the usual topology U, so that (R,U) is the Euclidean real topological space.

Let \( S = [0.1] \bigcup \{(1,2) \bigcap \mathbb{Q} \} \) where \( \mathbb{Q} \) stands for the set of rational numbers. Then S is b-open but neither semi-open nor pre-open.
On the other hand, let $T = [0,1] \cup \mathbb{Q}$. Then $T$ is semi-pre-open (i.e. $\beta$-open) but not $b$-open.

The classes of pre-open, semi-open, $\alpha$-open and semi-pre-open and $b$-open subsets of a space $(X,T)$ are usually denoted by $PO(X,T)$, $SO(X,T)$, $T_\alpha$, $SPO(X,T)$ and $BO(X,T)$ respectively. All of them are larger than $T$ and closed under forming arbitrary unions.

O. Njastad showed that $T_\alpha$ is a topology on $X$. In general, anyone of the other classes need not be a topology on $X$. However the intersection of a semi-open set and an open set is semi-open. The same holds for $PO(X,T)$ and $SPO(X,T)$, $BO(X,T)$.

In 1996, D. Andrijevic made the fundamental observation:

**Proposition (1.1):**

For every space $(X,T)$, $PO(X,T) \cup SO(X,T) \subseteq BO(X,T) \subseteq SPO(X,T)$ holds but none of these implications can be reversed.

The class of all semi-closed (resp. pre-closed, b-closed, semi-pre-closed) sets of a space $(X,T)$ is denoted by $SC(X,T)$ (resp. $PC(X,T), BC(X,T)$ & $SPC(X,T)$).

It is well known that:

- $\alpha$-closed $\Rightarrow$ semi-closed $\Rightarrow$ b-closed $\Rightarrow$ semi-pre-closed.

- $\cup \Rightarrow$ closed $\Rightarrow$ pre-closed.

The smallest semi-closed (resp. pre-closed, b-closed, semi-pre-closed) set containing $A \subseteq X$ is called the semi-closure (resp. pre-closure, b-closure, semi-pre-closure) of $A$ and is usually denoted by $scl(A)$ (resp. $pcl(A), bcl(A)$ and $spcl(A)$ or $\beta cl(A)$).

Dually, the largest semi-open (resp. pre-open, b-open, semi-pre-open) set contained in $A \subseteq X$ is called the semi-interior (resp. pre-interior, b-interior, semi-pre-interior) of $A$ and is usually denoted by $sint(A)$ (resp. $ pint(A), bint(A)$ and $spint(A)$ or $\beta int(A)$).

By $cl_\alpha$ and $int_\alpha$ we denote the closure and interior operator in $(X,T_\alpha)$.

The following two lemmas are concerned with the union of $b$-open sets & intersection of b-closed sets:

**Lemma (1.1):**

Arbitrary union of $b$-open sets is $b$-open.

**Proof:**

Let $\{A_\alpha\}_{\alpha \in \Delta}$ be a family of $b$-open sets in a space $(X,T)$, then

$$A_\alpha \subseteq cl(int(A_\alpha)) \cup int(cl(A_\alpha)), \forall \alpha \in \Delta$$

Now,

$$\bigcup_{\alpha \in \Delta} A_\alpha \subseteq \bigcup_{\alpha \in \Delta} \{cl(int(A_\alpha)) \cup int(cl(A_\alpha))\}$$

$$= [\bigcup_{\alpha \in \Delta} \{cl(int(A_\alpha))\} \cup [\bigcup_{\alpha \in \Delta} \{int(cl(A_\alpha))\}]$$

$$\subseteq [cl(\bigcup_{\alpha \in \Delta} (int(A_\alpha))) \cup [int(\bigcup_{\alpha \in \Delta} cl(A_\alpha))]$$

$$\subseteq [cl(\{int(\bigcup_{\alpha \in \Delta} A_\alpha)\}) \cup [int(\{cl(\bigcup_{\alpha \in \Delta} A_\alpha)\})]$$

$$\Rightarrow \bigcup_{\alpha \in \Delta} A_\alpha \text{ is also a } b \text{-open set.}$$

**Lemma (1.2):**

Arbitrary intersection of $b$-closed sets is $b$-closed.

**Proof:**

Let $\{B_\alpha\}_{\alpha \in \Delta}$ be a family of $b$-closed sets in a space $(X,T)$, then

$$cl(int(B_\alpha)) \cap int(cl(B_\alpha)) \subseteq B_\alpha, \forall \alpha \in \Delta.$$ 

Since, $\{B_\alpha^C\}_{\alpha \in \Delta}$ is an arbitrary indexed family of $b$-open sets, hence by lemma (1.1) $\bigcup_{\alpha \in \Delta} B_\alpha^C$ is a $b$-open set. But

$$\bigcup_{\alpha \in \Delta} B_\alpha^C = \left(\bigcap_{\alpha \in \Delta} B_\alpha\right)^C \Rightarrow \bigcap_{\alpha \in \Delta} B_\alpha \text{ is a } b \text{-closed set.}$$

**Note (1.1):** Above two lemmas ensure the existence of $b$-closure(bcl) and $b$-interior (bint) of a set in a topological space.
Of course bcl(A), being the smallest b-closed set containing A(S X), is the intersection of all b-closed subsets of X containing A, where (X,T) is a topological space.

And bint(A), being the largest b-open set contained in A(S X), is the union of all b-open subsets of X contained in A, where (X,T) is a topological space.

Now, we mention the following lemma which is useful in the sequel:

Lemma (1.3):
Let (X,T) be a topological space and A(S X), then
(a) \((T-bcl(A))^c = T-bint(A^C)\).
(b) \((T-bint(A))^c = T-bcl(A^C)\).

Proof: Let A(S X) where (X,T) is a topological space.
(a) Now, \(T-bcl(A) = \bigcap \{F: A \subseteq F \text{ and } F \text{ is T-b-closed set}\}\).
\([\text{CL}((T-bcl(A))^c) = \bigcap \{F: A \subseteq F \text{ and } F \text{ is T-b-closed set}\}]^c = \bigcup \{F^c: F^c \subseteq A^C \text{ and } F^c \text{ is T-b-open set}\} = T-bint(A^C)\).
(b) Similarly, \((T-bint(A))^c = T-bcl(A^C)\).

Remark (1.1):
The lemmas (1.1),(1.2)&(1.3) have been mentioned in the mathematical paper [3]. An extensive study of the operators scl, pcl, spcl, cl as well as sint, pint, spint, int_a was done by D.Andrijevic[1]. We recollect some of relations that, together with their duals, we shall use in the sequel.

Proposition (1.2): Let S be a subset of a space X. Then
(1) \(\text{cl}_a S = S \cup (\text{cl}(\text{int}(\text{cl}(S)))\),
\(\text{int}_a S = S \cap (\text{int}(\text{cl}(\text{int}(S)))\), &
\(\text{cl}_a \text{ (int}_a S) = \text{cl}(\text{int}(S)) \& \text{int}_a \text{ (cl}_a S) = \text{int}(\text{cl}(S))\).
(2) \(\text{scl} S = S \cup (\text{int}(\text{cl}(S)))\),
\(\text{sint} S = S \cap (\text{int}(\text{cl}(S)))\).
(3) \(\text{pcl} S = S \cup (\text{int}(\text{cl}(S)))\),
\(\text{pint} S = S \cap (\text{int}(\text{cl}(S)))\).
(4) \(\text{spcl} S = S \cup (\text{int}(\text{cl}(\text{int}(S))))\),
\(\text{spint} S = S \cap (\text{int}(\text{cl}(\text{int}(S))))\).

Proposition (1.3): Let S be a subset of a space X. Then
(1) \(\text{scl} \text{ (sint} S = \text{ sint} \cup (\text{int}(\text{cl}(\text{int}(S))))\),
(2) \(\text{pcl} \text{ (pint} S = \text{ pint} \cup (\text{int}(\text{cl}(\text{int}(S))))\),
(3) \(\text{spcl} \text{ (spint} S = \text{ spint} \cup (\text{int}(\text{cl}(\text{int}(S))))\).
(4) \(\text{int} \text{ (scl} S = \text{ pint} \cup (\text{int}(\text{cl}(\text{cl}(S))))\).
(5) \(\text{int} \text{ (pcl} S = \text{ cl} \cup (\text{int}(\text{cl}(\text{int}(S))))\).

Remark (1.2):
(a) It has been established in [1] that S is semi-pre-open iff
\(S \subseteq \text{sint(cl(cl(S)))}\).
(b) The condition \(S \subseteq \text{scl (sint(S))}\) characterizes the semi-open sets.
(c) The condition \(S \subseteq \text{pint(pcl(S))}\) characterizes the pre-open sets.

Here, \(S \subseteq X \& (X,T)\) is a topological space.

§2. Characterization of b-open sets:
We, now, prove the following theorem which characterizes a b-open set:

Theorem (2.1):
In a topological space (X,T), for a subset S of X, the following are equivalent:
(a) S is b-open.
(b) \(S = \text{ pint} S \cup \text{ sint} S\).
(c) \(S \subseteq \text{ pcl(pint}(S))\).

Proof:
Let S \(\subseteq X\) where (X,T) is a topological space.
(a) \(\Rightarrow\) (b):
Suppose that S is a b-open set. So, \(S \subseteq (\text{cl}(\text{int} S)) \cup (\text{int}(\text{cl} S))\).

Now,
pint \( S \cup \text{sint} S = \{\bigcap \text{int}(\text{cl } S)\} \cup \{\bigcup \text{cl}(\text{int } S)\}, \) [by Prop.(1.2)(2) & (3)].

\[ = S \bigcap \{\text{int}(\text{cl } S)\} \cup \{\text{cl}(\text{int } S)\}.
\]

\[ = S.\]

(b)\( \Rightarrow \) (c):

Let \( S = \text{pint } S \cup \text{sint } S, \) Then using proposition (1.2)(2)&(1.3)(2), we have

\[ S = \text{pint } S \cup \{\bigcap \text{cl}(\text{int } S)\}
\]

\[ \subseteq \text{pint } S \cup \text{cl}(\text{int } S)
\]

\[ = \text{pcl}(\text{pint } S).
\]

i.e. \( S \subseteq \text{pcl}(\text{pint } S).\)

(c)\( \Rightarrow \) (a):

Let \( S \subseteq \text{pcl}(\text{pint } S).\)

Then \( S \subseteq \text{pint } S \cup \text{cl}(\text{int } S)\) [Prop(1.3)(2)].

i.e. \( S \subseteq \{\bigcap \text{int}(\text{cl } S)\} \cup \{\text{cl}(\text{int } S)\} \), [Prop(1.2)(3)].

\[ = \{\bigcup \text{cl}(\text{int } S)\} \cap \{\text{int}(\text{cl } S)\} \cup \{\text{cl}(\text{int } S)\}.
\]

\[ \subseteq \text{int}(\text{cl } S) \cup \text{cl}(\text{int } S).
\]

i.e. \( S \) is a \( \text{b-open set.} \)

Hence, the theorem.

Note (2.1):

(a) It follows from theorem (2.3) (b) that every \( \text{b-open set} \) can be represented as a union of pre-open set and a semi-open set.

Also, \( \text{pint } S \cup \text{sint } S = \text{pint } S \{\bigcap \text{cl}(\text{int } S)\}.\)

\[ = \text{pint } S \{\text{clo}(\text{int } S)\}.
\]

This means that \( \text{pint } S \cup \text{sint } S \) is pre open. Hence, one can always have a disjoint union.

(b) If \( S \) be a \( \text{b-open set} such that \text{int } S = \emptyset, \) then

\[ \text{sint } S = \{\bigcap \text{cl}(\text{int } S)\} \]

This provides that \( \text{sint } S = \emptyset. \) Consequently, we have \( S = \text{pint } S \cup \text{sint } S = \text{pint } S \) i.e. \( S \) is a \( \text{pre-open set.} \)

Theorem (2.2):

If \( (X,T) \) is a topological space, then

(a) The intersection of an \( \alpha \)-open set and a \( \text{b-open set} \) is a \( \text{b-open set.} \)

(b) \( T \) and \( T_a \) have the same class of \( \text{b-open sets.} \)

Proof: Suppose that \( (X,T) \) is a topological space.

(a) Let \( A \) be an \( \alpha \)-set & \( B \), a \( \text{b-open set.} \)

Now, \( S = A \cap B \)

\[ = \text{int } A \cap \text{bint } B
\]

\[ \subseteq \text{sint } A \cap \text{bint } B
\]

\[ = \text{int } (A \cap B)
\]

\[ = \text{bint } (S)
\]

i.e. \( S \subseteq \text{bint } (S).\)

But \( \text{bint } (S) \subseteq S.\)

Hence, \( S = \text{bint } (S) \) i.e. \( S = A \cap B \) is a \( \text{b-open set.} \)

(b) Let \( S \) be an arbitrary \( \text{b-open set} \) w.r.t. \( T. \)

Then, \( S \subseteq \{\text{cl}(\text{int } (S))\} \cup \{\text{int } (\text{cl } (S))\} \) [Prop(1.2)(1)]

\[ \Rightarrow S \text{ is a \( \text{b-open set} \) w.r.t. } T_a.
\]

Thus, \( S \subseteq \text{BO}(X,T) \Rightarrow S \subseteq \text{BO}(X,T_a). \) Hence, the theorem.

Theorem (2.3):

If \( S \) be a subset of a space \( (X,T) \), then

(a) \( \text{bcl } S = \text{scI } S \cap \text{pcI } S.\)

(b) \( \text{bint } S = \text{sint } S \cup \text{pint } S.\)

Proof:

Let \( S \subseteq X \) where \( (X,T) \) is a topological space.

(a) Since \( \text{bcl } S \) is a \( \text{b-closed set,} \)

Hence, \( \text{int}(\text{cl}(\text{bcl } S)) \cup \text{cl}(\text{int}(\text{bcl } S)) \)

Again, \( \text{int}(\text{cl } S) \cap \text{cl}(\text{int } S) \subseteq \text{int}(\text{cl } S) \cap \text{cl}(\text{int}(\text{bcl } S)). \)
Theorem (2.4):

If \( S \) is a subset of a space \((X,T)\), then \( \text{bint}(\text{bcl}(S)) = \text{bcl}(\text{bint}(S)) \).

Proof:

Let \( S \) be a subset of a space \((X,T)\).

Now, \( \text{bint}(\text{bcl}(S)) = \text{sint}(\text{bcl}(S)) \cup \text{pint}(\text{bcl}(S)) \)
\[ = \text{bcl}(\text{sint}(S)) \cup \text{pint}(\text{bcl}(S)) \]
\[ = \text{scl}(\text{sint}(S)) \cup \text{pint}(\text{bcl}(S)) \] \[ \quad \text{(i)} \]

\( \text{bcl}(\text{bint}(S)) = \text{bcl}(\text{sint}(S)) \cup \text{pint}(\text{scl}(S)) \)
\[ = \text{bcl}(\text{sint}(S)) \cup \text{pint}(\text{bcl}(S)) \]
\[ = \text{scl}(\text{sint}(S)) \cup \text{pint}(\text{bcl}(S)) \] \[ \quad \text{\text{(ii)}} \]

Hence, from (i) & (ii), it follows that
\[ \text{bint}(S) = \text{sint}(S) \cup \text{pint}(S). \]

Hence, the theorem.

### Proposition (2.1):

Let \( S \) be a subset of a space \((X,T)\), then

1. \( \text{bcl}(\text{int}(S)) = \text{int}(\text{cl}(S)) \)
2. \( \text{bint}(\text{cl}(S)) = \text{cl}(\text{bint}(S)) \)
3. \( \text{bcl}(\text{sint}(S)) = \text{scl}(\text{sint}(S)) = \text{sint}(\text{bcl}(S)) \)
4. \( \text{bint}(\text{scl}(S)) = \text{sint}(\text{scl}(S)) \)
5. \( \text{scl}(\text{bint}(S)) = \text{scl}(\text{bint}(S)) \)
6. \( \text{scl}(\text{bint}(S)) = \text{scl}(\text{scl}(S)) \)
7. \( \text{pint}(\text{bcl}(S)) = \text{bcl}(\text{pint}(S)) = \text{pint}(\text{pcl}(S)) \)
8. \( \text{pcl}(\text{bint}(S)) = \text{pint}(\text{pcl}(S)) \)
9. \( \text{spcl}(\text{bint}(S)) = \text{pint}(\text{spcl}(S)) = \text{scl}(\text{scl}(S)) \)
10. \( \text{spcl}(\text{bint}(S)) = \text{pint}(\text{spcl}(S)) = \text{scl}(\text{scl}(S)) \)

Of course, these results enable us to relate the operators of b-closure and b-interior to the other operators defined by generalized open sets.

### § Some Characteristic Properties Of b-open sets:

The following theorems procure beforehand a unique property which characterizes b-open sets:

**Theorem (2.5):**

A subset \( A \) of a topological space \((X,T)\) is b-open if and only if every closed set \( F \) containing \( A \), there exists the union of maximal open set \( M \) contained in \( \text{cl}(A) \) and the minimal closed set \( N \) containing \( \text{int}(A) \) such that \( A \subseteq M \cup N \subseteq F \).

**Proof:**

Let \( A \) be a b-open set in a topological space \((X,T)\).
Then $A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$. (1)

Let $A \subseteq F$ and $F$ is closed so that $\text{cl}(A) \subseteq F$.

Let $M = \text{int}(\text{cl}(A))$, then $M$ is the maximal open set contained in $\text{cl}(A)$.

Let $N = \text{cl}(\text{int}(A))$, then $N$ is the minimal closed set containing $\text{int}(A)$.

Again, $A \subseteq \text{cl}(A) \subseteq F$ & $\text{int}(\text{cl}(A)) \subseteq \text{cl}(A)$.

$\text{int} (\text{cl}(A)) \subseteq F$. (2)

Next, $\text{int}(A) \subseteq \text{cl}(\text{int}(A)) \subseteq \text{cl}(A)$ & $\text{cl}(A) \subseteq F$.

$\text{cl}(\text{int}(A)) \subseteq F$. (3)

From (2) & (3), we have

$\text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A)) \subseteq F$. (4)

Combining (1) & (4), we have

$A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A)) \subseteq F$.

Or, $A \subseteq M \cup N \subseteq F$.

Conversely, assume that the condition holds good i.e.

$A \subseteq M \cup N \subseteq F$ where $A$ is a subset in a topological space, $F$ is closed & $M$ is the maximal open set contained in $\text{cl}(A)$, $N$ is the minimal closed set containing $\text{int}(A)$.

Therefore, $M = \text{int}(\text{cl}(A))$ & $N = \text{cl}(\text{int}(A))$.

Thus, the above condition reduces to $A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A)) \subseteq F$.

This means that $A$ is a $b$-open set.

Hence, the theorem.

**Theorem (2.6):**

A subset $A$ in a topological space $(X,T)$ is $b$-open if and only if there exists a pre-open set $U$ in $(X,T)$ such that $U \subseteq A \subseteq \text{pcl}(U)$.

**Proof:**

Let $A$ be a subset of $X$. Then by theorem (2.1),

$A \subseteq \text{pcl}(\text{pint}(A))$. (1)

Now, as usual $\text{pint}(A) \subseteq A$ and $U = \text{pint}(A)$ is a pre-open set. Hence, from (1) it follows that $U \subseteq A \subseteq \text{pcl}(U)$. Conversely, for a set $A$ there exists pre-open set $U$ such that

$U \subseteq A \subseteq \text{pcl}(U)$. (2)

Since $\text{pint}(A)$ is the maximal pre-open set contained in $A$, hence,

$U \subseteq \text{pint}(A) \subseteq A$. (3)

Now, $\text{pcl}(U) \subseteq \text{pcl}(\text{pint}(A))$. (4) [from (3)]

Combining (2) & (3) we get

$A \subseteq \text{pcl}(\text{pint}(A))$, which means that $A$ is a $b$-open set.

Hence, the theorem.

**Corollary (2.1):**

A subset $A$ in a topological space $(X,T)$ is $b$-open iff it contains pre-open set but not its pre-closure.

**Proof:** The proof is straightforward based on the above theorem, so omitted.

**Theorem (2.7):**

If $V$ is a $b$-open set in a space $(X,T)$, then $V \setminus \text{int}(\text{cl}(V))$ is a pre-open set.

**Proof:**

Let $V$ be a $b$-open set in a space $(X,T)$, then $V \subseteq \text{cl}(\text{int}(V)) \cup \text{int}(\text{cl}(V))$.

Since, int $A \subseteq A$ for all $A = X$, hence substituting $\text{cl}(\text{int}(V))$ for $A$, we have, int $(\text{cl}(\text{int}(V))) \subseteq \text{cl}(\text{int}(V))$.

This means that int $(\text{cl}(\text{int}(V)))$ is a semi-closed set. And in turn it is $b$-closed.

Now, $S = V – \text{int}(\text{cl}(V))$ is $b$-open.

Also, int $S = \emptyset$.

Using Note (2.1)(b), the above two facts provide that $S$ is a pre-open set. Hence, the theorem.
Conclusions:
Characterization and characteristic properties ensure that separation axioms can be framed in terms of b-open sets. Spaces profounded by b-open sets as b-connected and b-compact spaces are in existence. b-Hausdorff spaces & b-regular spaces are productive. Also different types of continuity evolve in the form of b-open sets.

References: