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Optimal Control based Time Optimal Low Thrust Orbit Raising

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Abstract

In the last few decades, electric propulsion systems have found use in science missions conducted by NASA, JAXA, and the ESA. [2] electric propulsion systems, such as an ion engine, are attractive options in mission design due to their high total ΔV to propellant mass ratios. Introductory material in orbital mechanics generally approaches orbital maneuvering with the assumption that maneuvers are performed by a spacecraft with high thrust propulsive capability. Orbit transfer techniques, including the Hohmann transfer, and Bi-Elliptic transfer, maintain the high thrust propulsion assumption by assuming that their orbit transfer maneuvers can be modeled as being impulsive in nature. Although, the approximation of high thrust ΔV manoeuvres are reasonable for spacecraft that performs its orbit transfer maneuvers with a chemical propulsion system, it fails to accurately approximate the behavior of a spacecraft that uses low thrust propulsion technology and continuous firing to maneuver from one orbit to another. In particular, although the Hohmann and Bi-Elliptic transfer techniques offer optimal orbit transfers assuming impulsive maneuvering, their assumptions entirely preclude their application to spacecraft using electric propulsion systems. This research investigates optimal control solutions to the continuous firing, low thrust orbit raising problem.

Keywords: Low Thrust, Optimal Control, Orbit Raising, Hohmann Transfer.

1. Introduction

Low thrust propulsion technologies have numerous advantages that designers have exploited for missions in Earth orbit, and for interplanetary missions. According to Keaton there are two major advantages that low-thrust propulsion systems have over conventional propulsion systems which are, in a strong gravitational field, low thrust propulsion systems require a fraction of the propellant mass than what conventional propulsion systems require, and, in a weak (interplanetary) gravitational field, low thrust systems can propel a spacecraft to higher velocities than a conventional system with a similar propellant mass. [3] Although, low thrust propulsion systems like ion engines are generally more efficient than conventional forms of propulsion, their low thrust requires them to burn continuously during orbit transfer maneuvers, hence complicating the orbit transfer problem.

The advantages afforded to space missions by electric propulsion have not been ignored by designers, and some consider low-thrust technologies to be the best option for sending interplanetary probes to distant locations within the solar systems. [4] NASA, JAXA, and ESA have all used electric propulsion for science missions, and the use of low thrust technologies has become routine over the last few decades. [2] Acceptance of low thrust technologies is based on their inherent advantages over conventional propulsion technologies; however, low-thrust technologies are incapable of the impulsive ΔV manoeuvres on which the Hohmann and Bi-Elliptic transfers are based. Due to the prevalence of low-thrust technologies in space, knowledge of how to design appropriate transfer trajectories for spacecraft that are incapable of impulsive ΔV manoeuvres are extremely valuable.

For spacecraft with the capability to perform the impulsive ΔV manoeuvres that traditional orbit raising techniques require, the Hohmann transfer often offers the fuel optimal orbit transfer solution. The Hohmann transfer often offers the fuel optimal the fuel optimal orbit transfer solution. The Hohmann transfer includes two impulsive ΔV manoeuvres: the first manoeuvre removes the spacecraft from its initial orbit and places it on an elliptical transfer trajectory that intersects the desired orbit at the transfer trajectory's apoapse. The second impulsive ΔV manoeuvre places the spacecraft into its final orbit. If the initial and final orbit are circular, the two ΔV manoeuvres are calculated as functions of parameters that describe the initial and final orbit. If the initial and final orbits are circular, the two ΔV manoeuvres are calculated as functions of parameters that describe the initial and final orbits. Equations (1)-(4) are used to define the ΔV required for the first maneuver.

$$a_{trans} = \frac{r_i + r_f}{2} \tag{1}$$

$$V_1 = \sqrt{\frac{\mu}{r_i}} \tag{2}$$

$$V_2 = \sqrt{\frac{2\mu}{r_i} - \frac{\mu}{a_{trans}}} \tag{3}$$

$$\Delta V_1 = \| V_2 - V_1 \| \tag{4}$$

Likewise, Equations (5)-(7) are used to define the ΔV required for the second maneuver.

$$V_3 = \sqrt{\frac{2\mu}{r_f} - \frac{\mu}{a_{trans}}} \tag{5}$$

$$V_4 = \sqrt{\frac{\mu}{r_f}} \tag{6}$$

$$\Delta V_2 = \| V_4 - V_3 \| \tag{7}$$

The transfer time required for a Hohmann transfer is half the period of the transfer orbit.
Determining the fuel optimal ΔV manoeuvres required to raise an orbit from an initial circular orbit is a trivial task when a spacecraft is capable of impulsive thrust maneuvers; however, determining the fuel optimal transfer maneuver for a spacecraft with a low thrust propulsion technology is much more involved.
Optimal control seeks to determine a control solution that minimizes a scalar cost function subjected to various constraints. Optimal control solutions can be found using direct methods that employ computers to directly calculate for the optimal control law or indirect methods which uses the calculus of variations to turn the optimal control problem into a boundary value problem. The most general optimal control cost function takes the form of Equation (8).

$$J = \phi\left(x\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} g\left(x, u, t\right).dt \tag{8}$$

where f represents a final state deviation cost and g represents a cost that is accumulated with time. Using the Lagrange multiplier method, constraints can be introduced to the cost function. For the orbit raising problem, the cost function is subject to constraints imposed by the dynamic system. The dynamic constraints are adjoined to the cost function using dynamic system. The dynamic constraints are adjoined to the cost function using dynamic Lagrange multipliers called adjoint states or co-states. The final augmented cost function for the orbit-raising problem takes the form shown in equation (9).

$$J_a\left(x, u, \lambda, t\right) = \phi\left(x\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} g\left(x, u, t\right).dt + \int_{t_0}^{t_f} \lambda^T \left(f - \dot{x}\right).dt \tag{9}$$

Analytical closed form control solution for low thrust orbit raising is given by Salvatore Alfano. [] The solution assumes orbit raising between two co-planar circular orbits and only considers tangential thrusting. [5] Press et al. discuss use of shooting methods for solving the two-point boundary value problems (TPBVP) which arise when using indirect methods for solving for optimal control solutions. [6] Optimal control is used to develop the baseline guidance for the second stage of the Crew Launch Vehicle and is derived from the Saturn V Iterative Guidance Mode. [7] The orbit raising optimal control problem and optimal control are derived in the next section.

2. METHODOLOGY

The problem of interest in this research is low thrust continuous firing orbit raising. Due to the ever-increasing prevalence of low thrust propulsion technologies in space, a time optimal control solution for low thrust spacecraft orbital maneuvering is vital. Various orbit transfers between circular orbits are analyzed. Indirect solution methods are employed to transform the optimal control problem into aa TPBVP. Also, MATLAB® built-in multiple shooting boundary value problem solver $bvp5c$ is used to solve for the co-state initial conditions that minimize the cost function.
The optimal control problem is formulated as follows: minimize the cost function, J, subject to the constraint that the

dynamic equations describing the motion of the spacecraft are satisfied at all times during the transfer. The cost function is developed so as to minimize the time of the orbital transfer. For the continuous firing case that is presented, minimizing the duration of the transfer is equivalent to minimizing the fuel used during the transfer. The cost function is presented in Equation (10).

$$J\left(x,u,t\right)=\int_{t_0}^{t_f}1.dt \tag{10}$$

The augmented cost function is formed by adjoining state constraints with Lagrange multipliers and is shown in Equation (11).

$$J_a\left(x,u,\lambda,t\right)=\int_{t_0}^{t_f}1.dt+\int_{t_0}^{t_f}\lambda^T\left(f-\dot{x}\right).dt \tag{11}$$

Defining the Hamiltonian operator, H, as

$$H\left(x,u,\lambda,t\right)=1+\lambda^T f \tag{12}$$

Altered cost function can be written as

$$J_a\left(x,u,\lambda,t\right)=\int_{t_0}^{t_f}H\left(x,u,\lambda,t\right).dt+\int_{t_0}^{t_f}\lambda^T\dot{x}.dt \tag{13}$$

To minimize the functional J_a, the first variation of J_a must be zero. The first variation of J_a is found using the calculus of variation as

$$\delta J_a\left(x,u,\lambda,t\right)=\int_{t_0}^{t_f}\frac{\partial H^T}{\partial x}.\delta x+\frac{\partial H^T}{\partial u}.\delta u+\frac{\partial H^T}{\partial \lambda}.\delta\lambda-\dot{x}^T.\delta\lambda+\lambda^T.\delta\dot{x}.dt+ [H-\lambda^T\dot{x}]|_{t_f}.dt_f-[H-\lambda^T\dot{x}]|_{t_0}.dt_0 \tag{14}$$

Integrating the final term in the integral by parts yields,

$$\delta J_a\left(x,u,\lambda,t\right)=\int_{t_0}^{t_f}\frac{\partial H^T}{\partial x}.\delta x+\frac{\partial H^T}{\partial u}.\delta u+\frac{\partial H^T}{\partial \lambda}.\delta\lambda-\dot{x}^T.\delta\lambda+\dot{\lambda}^T.\delta x.dt+ [H-\lambda^T\dot{x}]|_{t_f}.dt_f-[H-\lambda^T\dot{x}]|_{t_0}.dt_0-\lambda^T\delta x|_{t_f}+\lambda^T\delta x|_{t_0} \tag{15}$$

The problem, as formulated, has fixed initial conditions, a fixed initial time, fixed final conditions, and a free final time. Noting that

$$\delta x\left(t_f\right)=\delta x_f-\dot{x}\left(t_f\right).dt_f \tag{16}$$

$$dt_0=0 \tag{17}$$

$$\delta x\left(t_0\right)=0 \tag{18}$$

$$\delta x_f=0 \tag{19}$$

The variation of the altered cost function can be finally written as

$$\delta J_a\left(x,u,\lambda,t\right)=\int_{t_0}^{t_f}\left(\frac{\partial H^T}{\partial x}+\dot{\lambda}^T\right).\delta x+\frac{\partial H^T}{\partial u}.\delta u+\left(\frac{\partial H^T}{\partial \lambda}-\dot{x}^T\right).\delta\lambda.dt+H\left(t_f\right).dt_f \tag{20}$$

Since the variations on the right hand side of Equation (20) are arbitrary, their coefficients must vanish for the first variation of the cost function to be zero. These conditions imply the following

$$f=\dot{x} \tag{21}$$

$$\dot{\lambda}=-\frac{\partial H}{\partial x} \tag{22}$$

$$\frac{\partial H}{\partial u}=0 \tag{23}$$

$$H\left(t_f\right)=0 \tag{24}$$

To minimize J_a, the dynamic state constraints must be met, the co-state differential equations must be defined as the negative partial derivative of H with respect to the states, the partial derivative of H with respect to the control must be zero, and the Hamiltonian at the final time must be equal to zero.

The state equations used for this analysis represent two body motion in a plane in terms of polar coordinates, and are given in Equation (25).

$$\dot{x}=\begin{bmatrix}\dot{r}\\ \dot{s}\\ \dot{v}\end{bmatrix}=\begin{bmatrix} s \\ \frac{v^2}{r}-\frac{\mu}{r^2}+T\sin\left(\phi_c\right) \\ -\frac{sv}{r}+T\cos\left(\phi_c\right)\end{bmatrix} \tag{25}$$

where r, s, and v represent the orbital radius, radial velocity, and tangential velocity, respectively. ϕ_c is the control variable, an angle measured anti-parallel to the velocity vector. To simplify the analysis, the thrust acceleration, T, of the spacecraft with an electric engine is assumed to be constant. This assumption is reasonable given the exceptionally small mass flow rate of electric engines and the relatively small amount of time the engine is firing during the orbit raising maneuver.

Given these state equations, the Hamiltonian operator can be formed as

$$H\left(x,u,\lambda,t\right)=1+\lambda_r s+\lambda_s\left(\frac{v^2}{r}-\frac{\mu}{r^2}+T\sin\left(\phi_c\right)\right)+\lambda_v\left(-\frac{sv}{r}+T\cos\left(\phi_c\right)\right) \tag{26}$$

The co-state differential equations can be formed using Equation (22)

$$\dot{\lambda}=\begin{bmatrix} -\lambda_s\left(-\frac{v^2}{r^2}+\frac{2\mu}{r^3}\right)-\lambda_v\frac{sv}{r^2} \\ -\lambda_r+-\lambda_v\frac{v}{r} \\ -2\lambda_s\frac{v}{r}+\lambda_v\frac{s}{r}\end{bmatrix} \tag{27}$$

Using Equation (23), the optimal control law can be formed as a function of the co-states

$$\tan\left(\phi_c\right)=\frac{-\lambda_s}{-\lambda_v} \tag{28}$$

Where the signs of the co-states have been chosen to ensure that a time-optimal solution is found.

The optimal control law derived in Equation (28) can now be substituted back into Equation (25). The state and co-state equations are now coupled, and the TPBVP is completely formulated. The initial time is established to be zero. The initial states are fixed. The final state conditions are fixed as states for a spacecraft in a circular orbit. The final time is free, and the initial and final conditions for the co-states are unknown. The optimal control problem will be solved when the final time and initial and final co-state conditions are known.

MATLAB® built-in boundary value problem solver, *bvp5c*, is used to solve for the final time and initial conditions for the co-states. This solver is limited to boundary value problems with a known final time. This limitation can be worked around by appending a final time "state" to the differential equations and modifying the equations slightly so that *bvp5c* will solve for the final time as well as the co-state initial conditions. The modification is given in [9] and proceeds as follows:

$$\begin{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} \\ \lambda \\ \dot{t}_f \\ \end{bmatrix} = t_f \begin{bmatrix} f \\ g \\ 0 \end{bmatrix} \qquad 0 \le t \le 1 \tag{29}$$

where g represents the co-state differential equations. Using this modified system, $bvp5c$ solves for all unknowns in the boundary value problem. Boundary value problems are notoriously difficult to solve even if there is a MATLAB® built-in to do so. Convergence on a solution is heavily dependent on the initial guess for the initial conditions for the co-state and final time.

3. RESULTS

The first orbit-raising maneuver for which control is solved is a 300 Km increase in semi-major axis from an initial circular orbit of 7000 Km to a final circular orbit of 7300 Km. The spacecraft and orbital parameters used in all analyses are included in Table 1. The known and unknown states for this maneuver are included in Table 2.

Table 1: Spacecraft and Thruster Properties

Parameter	Value
Mass (Kg)	1000
Thrust Acceleration (Km/s^2)	$5 X 10^9$

Table 2: Initial and Final Conditions for a 300 Km Orbit-Raising Maneuver

State	Initial Time	Final Time
r (Km)	7000	7300
s (Km/s)	0	0
v (Km/s)	7.54605	7.38937

MATLAB® built-in boundary value solver converged to a solution after numerous iterations. The time required for the optimal transfer is found to be 31567 seconds. A spacecraft using impulsive manoeuvres and a Hohmann transfer takes approximately 3000 seconds to complete the same orbit-raising manoeuvre. The benefit of an electric engine is its fuel efficiency. The mass flow rate of an engine is given by Equation (30).

$$\dot{m} = \frac{\tau}{I_{sp} g_0} \tag{30}$$

Given an approximate thrust value of 5000 mN and a specific impulse of 7000 seconds, the electric propulsion powered spacecraft would only consume approximately 2 kg of fuel during the entire transfer. The superior efficiency of electric engines vastly outweighs the relative inefficiency of continuous thrusting.

The optimal control for a second, larger orbit transfer was calculated. The initial and final conditions for the second orbit-raising maneuver are included in Table 3.

Table 3: Initial and Final Conditions for an Orbit-Raising Maneuver from 10000 Km to 12540 Km

State	Initial Time	Final Time
r (Km)	11000	12540
s (Km/s)	0	0
v (Km/s)	6.01967	5.63794

The time required for the second manoeuvre is determined to be 76389 seconds. An equivalent Hohmann transfer would require 6354 seconds. The fuel consumption across the entire manoeuvre is only 5.6 Kg. The reduced fuel requirements for a spacecraft with electric propulsion allows for a cheaper orbital insertion or increased payload. The higher maximum velocity afforded by electric propulsion makes it particularly attractive for interplanetary missions.

It is interesting to note that, although the second orbit-raising manoeuvre raised the orbit by over five times the amount of the first orbit-raising manoeuvre, it takes only a little more than twice as long to complete. This is likely due to the increase effect of gravity across the first manoeuvre.

4. CONCLUSIONS

The massive fuel mass reduction that electric engines afford to designers vastly outweighs the cost of increased trajectory design complexity. The manual trajectory design process implemented in this research is extremely tedious and could be greatly improved given time. Currently, final orbit radii are increased in small increments and initial co-state, and time guesses are based on the initial co-states and final time estimated by $bvp5c$. The process is time consuming, and the initial co-states are nonlinear with respect to final orbit radius. As a result, the orbit-raising solution process slowed down significantly as the final orbit radius of 12540 km was approached. However, the solution process is systematic, and this optimal control algorithm could be used to place a spacecraft into any orbit that is desired.

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