# Engineering Applications of Differential equations

**B. Sumithra**

Department of Mathematics, Anna University, BIT Campus, Tiruchirappalli - 620 024, Tamilnadu, India.

**ABSTRACT**

***In this paper, the relevance of differential equations in engineering through their applications in various engineering disciplines and various types of differential equations are motivated by engineering applications; theory and techniques for solving differential equations are applied to solve practical engineering problems.***

**Keywords:** Differential equations, Applications, Partial differential equation, Heat equation.

## 1. INTRODUCTION

The Differential equations have wide applications in various engineering and science disciplines. In general, modeling of the variation of a physical quantity, such as temperature, pressure, displacement, velocity, stress, strain, current, voltage, or concentration of a pollutant, with the change of time or location, or both would result in differential equations. Similarly, studying the variation of some physical quantities on other physical quantities would also lead to differential equations. In fact, many engineering subjects, such as mechanical vibration or structural dynamics, heat transfer, or theory of electric circuits, are founded on the theory of differential equations. It is practically important for engineers to be able to model physical problems using mathematical equations, and then solve these equations so that the behavior of the systems concerned can be studied.

## 2. MOTIVATING EXAMPLES

It is important for engineers to be able to model physical problems using mathematical equations, and then solve these equations so that the behaviour of the systems concerned can be studied. In this section, a few examples are presented to illustrate how practical problems are modeled mathematically and how differential equations arise in them.

### 2.1 Motivating example-1

A tank contains a liquid of volume V(t), which is polluted with a pollutant concentration in percentage of c(t) at time t. To reduce the pollutant concentration, an inflow of rate $Q_{in}$ is injected to the tank. Unfortunately, the inflow is also polluted but to a lesser degree with a pollutant concentration $c_{in}$. It is assumed that the inflow is perfectly mixed with the liquid in the tank instantaneously. An outflow of rate $Q_{out}$ is removed from the tank. Suppose that, at time t = 0, the volume of the liquid is $V_0$ with a pollutant concentration of $c_0$.The equation governing the pollutant concentration c(t) is given by

$$[V_0 + (Q_{in} - Q_{out})t]\ (dc(t)/dt) + Q_{in}\, c(t) = Q_{in}\, c_{in};$$

with initial condition $c(0) = c_0$. This is a first-order ordinary differential equation.

### 2.2. Motivating example-2

Consider the suspension bridge, which consists of the main cable, the hangers, and the deck. The self-weight of the deck and the loads applied on the deck are transferred to the cable through the hangers. Set up the Cartesian coordinate system by placing the origin O at the lowest point of the cable. The cable can be modeled as subjected to a distributed load w(x). The equation governing the shape of the cable is given by

$$(d^2y/dx^2) = (w(x)/H)$$

;

where H is the tension in the cable at the lowest point O. This is a second-order ordinary differential equation.

**2.3. Motivating example-3**

Consider the vibration of a single-story shear building under the excitation of earthquake. The shear building consists of a rigid girder of mass m supported by columns of combined stiffness k. The vibration of the girder can be described by the horizontal displacement x(t). The earthquake is modeled by the displacement of the ground $x_0(t)$ as shown. When the girder vibrates, there is a damping force due to the internal friction between various components of the building, given by

$$c\,[x'(t) - x'_0(t)];$$

where c is the damping coefficient.

The relative displacement $y(t) = x(t) - x_0(t)$ between the girder and the ground is governed by the equation

$$my''(t) + cy'(t) + k\,y(t) = -mx_0''(t),$$

which is a second-order linear ordinary differential equation.

## 3. APPLICATIONS AND CONNECTIONS TO OTHER AREAS

Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behaviour of complex systems. The mathematical theory of differential equations first developed together with the sciences where the equations had originated and where the results found application. However, diverse problems, sometimes originating in quite distinct scientific fields, may give rise to identical differential equations. Whenever this happens, mathematical theory behind the equations can be viewed as a unifying principle behind diverse phenomena. As an example, consider propagation of light and sound in the atmosphere, and of waves on the surface of a pond. All of them may be described by the same second-order partial differential equation, the wave equation, which allows us to think of light and sound as forms of waves, much like familiar waves in the water. Conduction of heat, the theory of which was developed by Joseph Fourier, is governed by another second-order partial differential equation, the heat equation. It turns out that many diffusion processes, while seemingly different, are described by the same equation; the Black-Scholes equation in finance is, for instance, related to the heat equation.

**3.1In physics**

**1) Classical mechanics:**

So long as the force acting on a particle is known, Newton's second law is sufficient to describe the motion of a particle. Once independent relations for each force acting on a particle are available, they can be substituted into Newton's second law to obtain an ordinary differential equation, which is called the equation of motion.

**2) Electrodynamics:**

Maxwell's equations are a set of partial differential equations that, together with the Lorentz force law, form the foundation of classical electrodynamics, classical optics, and electric circuits. These fields in turn underlie modern electrical and communications technologies. Maxwell's equations describe how electric and magnetic fields are generated and altered by each other and by charges and currents. They are named after the Scottish physicist and mathematician James Clerk Maxwell, who published an early form of those equations between 1861 and 1862.

**3) General relativity:**

The Einstein field equations (EFE; also known as "Einstein's equations") are a set of ten partial differential equations in Albert Einstein's general theory of relativity which describe the fundamental interaction of gravitation as a result of space time being curved by matter and energy. First published by Einstein in 1915 as a tensor equation, the EFE equate local space time curvature (expressed by the Einstein tensor) with the local energy and momentum within that space time (expressed by the stress-energy tensor).

**4) Quantum mechanics:**

In quantum mechanics, the analogue of Newton's law is Schrdinger's equation (a partial differential equation) for a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localized). It is not a simple algebraic equation, but in general a linear partial differential equation, describing the time-evolution of the system's wave function (also called a "state function").

**5) Other important equations:**
Euler-Lagrange equation in classical mechanics, Hamilton's equations in classical mechanics, Radioactive decay in nuclear physics, Newton's law of cooling in thermodynamics, the wave equation, the heat equation in thermodynamics, Laplace's equation, which defines harmonic functions, Poisson's equation, the geodesic equation, the Navier-Stokes equations in fluid dynamics, the Diffusion equation in stochastic processes, the Convection-diffusion equation in fluid dynamics, the Cauchy-Riemann equations in complex analysis, the Poisson-Boltzmann equation in molecular dynamics, the shallow water equations, Universal differential equation and the Lorenz equations whose solutions exhibit chaotic flow.

### 3.2In biology

1) **Predator-prey equations:** The Lotka-Volterra equations, also known as the predator-prey equations, are a pair of first-order, non-linear, differential equations frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey.
2) **Other important equations**: Verhulst equation **-** biological population growth, von Bertalanffy model **-** biological individual growth, Replicator dynamics **-** found in theoretical biology and Hodgkin-Huxley model **-** neural action potentials.

### 3.3 In chemistry

Rate equation: The rate law or rate equation for a chemical reaction is a differential equation that links the reaction rate with concentrations or pressures of reactants and constant parameters (normally rate coefficients and partial reaction orders). To determine the rate equation for a particular system one combines the reaction rate with a mass balance for the system.

### 3.4 In economics

Important equations: The Black-Scholes Partial Differential Equation, Exogenous growth model, Malthusian growth model and the Vidale-Wolfe advertising model.

## 4. APPLICATION OF DIFFERENTIAL EQUATION IN FALLING OBJECT

An object is dropped from a height at time t = 0. If h(t) is the height of the object at time t, a(t) the acceleration and v(t) the velocity. The relationships between a, v and h are as follows:

$$a(t) = dv/dt; \qquad v(t) = dh/dt:$$

For a falling object, a(t) is constant and is equal to g = -9.8 m/s. Combining the above differential equations, we can easily deduce the following equation

$$d^2h/dt^2 = g$$

Integrate both sides of the above equation to obtain

$$dh/dt = gt + v_0$$

Integrate one more time to obtain

$$h(t) = (1/2)gt^2 + v_0t + h_0$$

The above equation describes the height of a falling object, from an initial height $h_0$ at an initial velocity $v_0$, as a function of time.

## 5. APPLICATION OF DIFFERENTIAL EQUATION IN NEWTON'S LAW OF COOLING

It is a model that describes, mathematically, the change in temperature of an object in a given environment. The law states that the rate of change (in time) of the temperature is proportional to the difference between the temperature T of the object and the temperature Te of the environment surrounding the object.

$$dT/dt = -k(T - Te)$$

Let $x = T - Te$ so that $dx / dt = dT / dt$.

Using the above change of variable, the above differential equation becomes

$$dx / dt = - kx$$

The solution to the above differential equation is given by

$$x = Ae - kt$$

substitute x by T–Te

$$T-Te = Ae - kt$$

Assume that at t=0 the temperature $T = T_0$

$$T_0-Te = Ae_0$$

Which gives $A = T_0 - Te$

The final expression for T(t) is given by

$$T(t) = Te + (T_0 - Te)e^{-kt}$$

This last expression shows how the temperature T of the object changes with time.

**5.1 Cooling/Warming law**

The mathematical formulation of Newton's empirical law of cooling of an object is given by the linear first-order differential equation

$$dT/dt = \alpha (T - T_m)$$

This is a separable differential equation. We have

$$dT/(T - T_m) = \alpha\, dt \quad \text{Or } T(t) = T_m + c_2e^{\alpha t}$$

**5.2 Population Growth and Decay**

In the differential equation

$$dN(t)/dt = k N(t)$$

where N(t) denotes population at time t and k is a constant of proportionality, serves as a model for population growth and decay of insects, animals and human population at certain places and duration.

Solution of this equation is

$N(t) = Ce^{kt}$, where C is the constant of integration:

$$dN(t)/N(t) = kdt$$

Integrating both sides we get

$$\ln N(t) = kt + \ln C \quad \text{or} \quad N(t) = C e^{kt} ,$$

C can be determined if N(t) is given at certain time.

**5.3 Radio-active Decay and Carbon Dating**

A radioactive substance decomposes at a rate proportional to its mass. This rate is called the decay rate. If m(t) represents the mass of a substance at any time, then the decay rate dm/dt is proportional to m(t). Let us recall that the half-life of a substance is the amount of time for it to decay to one-half of its initial mass.

**5.4 Carbon Dating**

The key to the carbon dating of paintings and other materials such as fossils and rocks lies in the phenomenon of radioactivity discovered at the turn of the century. The physicist Rutherford and his colleagues showed that the atoms of certain radioactive elements are unstable and that within a given time period a fixed portion of the atoms spontaneously disintegrate to form atoms of a new element. Because radioactivity is a property of the atom, Rutherford showed that the radioactivity of a substance is directly proportional to the number of atoms of the substance present. Thus, if N(t) denotes the number of atoms present at time t, then dN/dt, the number of atoms that disintegrate per unit time, is proportional to N; that is,

$$dN/dt = -\lambda N$$

**5.5 Drug Distribution (Concentration) in Human Body:**

To combat the infection to human a body appropriate dose of medicine is essential. Because the amount of the drug in the human body decreases with time medicine must be given in multiple doses. The rate at which the level y of the drug in a patient's blood decays can be modeled by the decay equation

$$dy/dt = -ky$$

where k is a constant to be experimentally determined for each drug. If initially, that is, at t = 0 a patient is given an initial dose $y_p$, then the drug level y at any time t is the solution of the above differential equations, that is,

$$y(t) = y_p e^{-kt}$$

## 6. CONCLUSION

In this paper a detailed analysis is presented to model the engineering problems to have a balance between theory and applications using differential equations.

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