In this paper the M/M/2/∞ queueing model with controllable arrival rates and feedback is considered. Distribution of busy period and mean length of busy period carried out for this model. The analytical results are numerically illustrated and the effect of the nodal parameters on the expected length of busy period is studied and relevant conclusions are presented.

Keywords: Two server Markovian queueing model, Controllable arrival rates, Feedback, infinite capacity, Busy period Analysis.

1. Introduction
In the earlier work, Thiagarajan and Srinivasan [6] have analysed the busy period analysis of interdependent queueing model with controllable arrival rates are employed and obtained the average length of the busy period for the model. Takagi and Tarabia [4] have studied an explicit probability density function for the length of a busy period for a finite capacity. Artalejo and Lopez Herrero [1] have analysed the busy period of the retrial queue with general service time distribution. Gopalan [2] has analysed the busy period analysis of a two-stage multi-product system. Soren Asmussen [7] has studied a busy period analysis, rare Events and Transient behaviour in fluid flow Models. Thangaraj and Vanitha [5] have analysed on the analysis of M/M/1 feedback queue with catastrophe using continued fractions. In this Chapter, busy period analysis of two server M/M/2/∞ interdependent queueing model with controllable arrival rates and feedback is considered. In section 2, the description of the queueing model is given stating the relevant postulates. In section 3, Differential-Difference Equations are derived. In section 4, Busy period analysis of the model and average length of the busy period of the model is obtained. In section 5, the analytical results are numerically illustrated and relevant conclusion is presented based upon a hypothetical data.

2. Description of the Model
Consider two server infinite capacity queueing system with controllable arrival rates and feedback. Customers arrive at the service station one by one according to a bivariate Poisson stream with arrival rates \((\lambda_0, \lambda_1)\) (>0). There are two servers which provides service to all the arriving customers. Service times are identically and independently distributed exponential random variables with mean rate \((\mu)\) (>0). After the completion of each service, the customers can either join at the end of the queue with probability \(p\) or customers can leave the system with probability \(q\), \(p+q=1\). The customer both newly arrived and those who opted for feedback are served in the order in which they join the tail of the original queue. It is assumed that there is no difference between regular arrivals and feedback arrivals. The customers are served according to the first come first served rule with a busy period as the interval of time from the instant customers arrive either with feedback or without feedback at an empty system and their service begins to the instant when the server becomes free for the first time. It is assumed that, The arrival process \(\{X_1(t)\}\) and the service process \(\{X_2(t)\}\) of the system are correlated and follow a bivariate Poisson distribution is given by

\[
P[X_1(t) = x_1, X_2(t) = x_2] = \frac{e^{-(\lambda_0 + \mu + \epsilon_t)} \sum_{j=0}^{\min(x_1, x_2)} \left(\epsilon_t\right)^j \left((\lambda_1 - \epsilon) t\right)^{x_1 - j} \left((\mu - \epsilon) t\right)^{x_2 - j}}{j!(x_1 - j)!(x_2 - j)!}
\]
where, $x_0, x_1 = 1, 2, ..., \text{and} \lambda_0, \lambda_1, \mu > 0, 0 \leq \varepsilon < \min (\lambda_i \mu), (i=0,1)$ with parameters $\lambda_0, \lambda_1, \mu$ and $\varepsilon$ for mean faster rate of arrivals, mean slower rate of arrivals either with feedback or without feedback, identical mean service rate for first server, second server and mean dependence rate respectively. Also the state space of the system is

\[ s = \{1, 2, 3, ..., r-1, r, r+1, ..., R-1, R, R+1, ..., \} \]

Postulates of the model are

1. Probability that there is no arrival and no service completion during a small interval of time $h$, when the system is in faster rate of arrivals either with feedback or without feedback, is

\[ 1 - \left( \lambda_0 - \varepsilon \right) + 2p(\mu - \varepsilon) + 2q(\mu - \varepsilon) \right] h + o(h) \]

2. Probability that there is one arrival and no service completion during a small interval of time $h$, when the system is in faster rate of arrivals either with feedback or without feedback, is

\[ (\lambda_1 - \varepsilon) h + o(h) \]

3. Probability that there is no arrival and no service completion during a small interval of time $h$, when the system is in slower rate of arrivals either with feedback or without feedback, is

\[ 1 - \left( \lambda_1 - \varepsilon \right) + 2p(\mu - \varepsilon) + 2q(\mu - \varepsilon) \right] h + o(h) \]

4. Probability that there is one arrival and no service completion during a small interval of time $h$, when the system is in slower rate of arrivals either with feedback or without feedback is

\[ (\lambda_1 - \varepsilon) h + o(h) \]

5. Probability that there is no arrival and one service completion during a small interval of time $h$, when the system is either in faster or is in slower rate of arrivals either with feedback or without feedback is

\[ 2p(\mu - \varepsilon) + 2q(\mu - \varepsilon) \right] h + o(h) \]

6. Probability that there is one arrival and one service completion during a small interval of time $h$, when the system is either in faster or slower rate of arrivals either with feedback or without feedback is

\[ \left( \lambda_0 - \varepsilon \right) + \left( \lambda_1 - \varepsilon \right) + 2p(\mu - \varepsilon) + 2q(\mu - \varepsilon) \right] h + o(h) \]

3. Difference - Difference Equations

\[ P_n(t_0): \] the probability that there are $n$ customers in the system at time $t_0$ when the system is in faster rate of arrivals either with feedback or without feedback.

\[ P_n(t_1): \] the probability that there are $n$ customers in the system at time $t_1$ when the system is in slower rate of arrivals either with feedback or without feedback.

We observe that $P_n(t_0)$ exists when $n = 0, 1, 2, 3, ..., r-1$; both $P_n(t_0)$ and $P_n(t_1)$ exist when $n = r+1, r+2, r+3, ..., R-2, R-1$; only $P_1(t_1)$ exists when $n = R, R+1, R+2, ...$. Assume that the initial system size when the system is in faster rate of arrivals is $1$ either with feedback or without feedback and that of when the system is in slower rate of arrivals is $r+1$.

Let $P_0(t_0)$ be the busy period density when the system is in faster rate of arrivals either with feedback or without feedback and $P_r(t_1)$ be the busy period density when the system is in slower rate of arrivals either with feedback or without feedback.

The differential-difference equations governing the system size with an absorbing barrier imposed at zero system size when the system is in faster rate of arrivals either with feedback or without feedback and with an absorbing barrier imposed at $(r+1)$ system size when the system is in slower rate of arrivals either with feedback or without feedback, are

\[ P'_0(t_0) = -2(\mu - \varepsilon) P_1(t_0) \quad \text{(because of the absorbing barrier)} \quad \text{... (3.1)} \]

\[ P'_1(t_0) = -\left( \lambda_0 - \varepsilon \right) + 2q(\mu - \varepsilon) \right] P_1(t_0) + 2q(\mu - \varepsilon) P_2(t_0) \quad \text{... (3.2)} \]

\[ P'_n(t_0) = -\left( \lambda_0 - \varepsilon \right) + 2q(\mu - \varepsilon) \right] P_n(t_0) + 2q(\mu - \varepsilon) P_{n+1}(t_0) \]

\[ + (\lambda_0 - \varepsilon) P_{n-1}(t_0) \quad \text{for} \quad n = 2, 3, 4, ..., r-1 \quad \text{... (3.3)} \]
Let \( \rho(0) \) = \( \frac{\lambda_0 - \epsilon}{2q(\mu - \epsilon)} \) and \( \rho(1) \) = \( \frac{\lambda_1 - \epsilon}{2q(\mu - \epsilon)} \)

where \( \frac{\rho(0)}{2} \) is faster rate of arrivals intensity and \( \frac{\rho(1)}{2} \) is slower rate of arrivals intensity.

4. Busy Period Analysis of the Model

Define \( P(z, t_0) = \sum_{n=0}^{R-1} P_n(t_0)z^n \) ... (4.1)

\[ P(z, t_1) = \sum_{n=R+1}^{\infty} P_n(t_1)z^n \] ... (4.2)

Such that the summation is convergent in and on unit circle. When (3.1) to (3.6) are multiplied through by \( z^n \) for \( n = 0, 1, 2, ..., R-1 \) and summing over from \( n = 0, 1, 2, ..., R-1 \), it is found that

\[
\frac{d}{dt_0} [P(z, t_0)] = \sum_{n=0}^{R-1} P_n(t_0)z^n
\]

\[
= \left\{ -[(\lambda_0 - \epsilon) + 2q(\mu - \epsilon)]\sum_{n=0}^{R-1} P_n(t_0) + (\lambda_0 - \epsilon)\sum_{n=0}^{R-1} P_{n-1}(t_0) + 2q(\mu - \epsilon)\sum_{n=0}^{R-1} P_{n+1}(0) \right\}z^n
\]

\[
+ 2q(\mu - \epsilon)P_{r+1}(t_1)z^{r+1} + 2q(\mu - \epsilon)P_{r+1}(t_0)z^{r+1}
\]

\[
+ \left\{ -(\lambda_0 - \epsilon) + [(\lambda_0 - \epsilon) + 2q(\mu - \epsilon)]z - 2q\frac{z(\mu - \epsilon)}{z} \right\}P_0(0)
\]

\[
= -[(\lambda_0 - \epsilon) + 2q(\mu - \epsilon)]\sum_{n=0}^{R-1} P_n(t_0)z^n + (\lambda_0 - \epsilon)z\sum_{n=1}^{R-1} P_{n-1}(t_0)z^{n-1}
\]
\[ + \frac{2q(\mu - \varepsilon)}{z} \sum_{n=0}^{\infty} P_{n+1}(t_0) z^{n+1} + 2q(\mu - \varepsilon) P_{r+1}(t_1) z^{r+1} \]
\[ + \left\{ -\left(\lambda_0 - \varepsilon\right)z + \left[\left(\lambda_0 - \varepsilon\right) + 2q(\mu - \varepsilon)\right] z - 2q\left(\frac{\mu - \varepsilon}{z}\right)\right\} P_0(t_0) \]
\[ z^d \frac{d}{dt_0} [P(z,t_o)] = \left[\left(\lambda_0 - \varepsilon\right) z^2 - \left[\left(\lambda_0 - \varepsilon\right) + 2q(\mu - \varepsilon)\right] z + 2q(\mu - \varepsilon)\right] P(z,t_o) \]
\[ + \left[ -\left(\lambda_0 - \varepsilon\right) z^2 + \left[\left(\lambda_0 - \varepsilon\right) + 2q(\mu - \varepsilon)\right] z - 2q(\mu - \varepsilon)\right] P_0(t_0) \]
\[ + 2q(\mu - \varepsilon) z^{r+1} P_{r+1}(t_1) \quad \ldots (4.3) \]

Taking Laplace Transform on both sides of (4.3) we get
\[ z^2 \left[ s_o P(z,s_o) - P(z,0) \right] = \left[\left(\lambda_0 - \varepsilon\right) z^2 - \left[\left(\lambda_0 - \varepsilon\right) + 2q(\mu - \varepsilon)\right] z + 2q(\mu - \varepsilon)\right] \bar{P}(z,s_o) \]
\[ + \left[ -\left(\lambda_0 - \varepsilon\right) z^2 + \left[\left(\lambda_0 - \varepsilon\right) + 2q(\mu - \varepsilon)\right] z - 2q(\mu - \varepsilon)\right] \bar{P}_0(s_o) \]
\[ - 2q(\mu - \varepsilon) \bar{P}_0(s_o) + 2q(\mu - \varepsilon) z^{r+1} \bar{P}_{r+1}(s_i) \]
\[ \bar{P}(z,s_o) = \frac{z^2 - (1-z)\left[ 2q(\mu - \varepsilon) - (\lambda_0 - \varepsilon)z \right] \bar{P}_0(s_o) + 2q(\mu - \varepsilon) z^{r+1} \bar{P}_{r+1}(s_i)}{\left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right] z - 2q(\mu - \varepsilon) - (\lambda_0 - \varepsilon) z^2} \]

where
\[ L\{P(z,t_0)\} = \bar{P}(z,s_o) \quad \ldots (4.4) \]
\[ L\{P_0(t_0)\} = \bar{P}_0(s_o), L\{P_{r+1}(t_1)\} = \bar{P}_{r+1}(s_i) \quad \text{and} \quad P(z,0) = z \]

Since the Laplace Transform \( \bar{P}(z,s_o) \) converges in the region \( |z| \leq 1 \), \( \text{Real} \ (s_o) \geq 0 \), wherever the denominator of the right hand side of (4.4) has zeros in that region, so must the numerator, the denominator has two zeros, that is
\[ (\lambda_0 - \varepsilon) z^2 - \left[\left(\lambda_0 - \varepsilon\right) + 2q(\mu - \varepsilon) + s_o \right] z + 2q(\mu - \varepsilon) = 0 \]
\[ Z_{0}^{(0)} = \left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right] - \sqrt{\left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right]^2 - 4(\lambda_0 - \varepsilon)2q(\mu - \varepsilon)} \]
\[ 2(\lambda_0 - \varepsilon) \]
\[ Z_{0}^{(1)} = \left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right] - \sqrt{\left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right]^2 - 8(\lambda_0 - \varepsilon)q(\mu - \varepsilon)} \]
\[ 2(\lambda_0 - \varepsilon) \]
\[ Z_{2}^{(0)} = \left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right] + \sqrt{\left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right]^2 - 4(\lambda_0 - \varepsilon)2q(\mu - \varepsilon)} \]
\[ 2(\lambda_0 - \varepsilon) \]
\[ Z_{2}^{(1)} = \left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right] + \sqrt{\left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right]^2 - 8(\lambda_0 - \varepsilon)q(\mu - \varepsilon)} \]
\[ 2(\lambda_0 - \varepsilon) \]

and hence
\[ Z_{0}^{(0)} = \left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right] - \sqrt{\left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right]^2 - 8(\lambda_0 - \varepsilon)q(\mu - \varepsilon)} \]
\[ 2(\lambda_0 - \varepsilon) \]
\[ Z_{0}^{(1)} = \left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right] + \sqrt{\left[ (\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_o \right]^2 - 8(\lambda_0 - \varepsilon)q(\mu - \varepsilon)} \]
\[ 2(\lambda_0 - \varepsilon) \]
\[ \ldots (4.5) \]
From (4.5), it is clear that \[ z_1^{(0)} < z_2^{(0)} \]

\[ z_1^{(0)} + z_2^{(0)} = \frac{2(\lambda_0 - \varepsilon) + 4q(\mu - \varepsilon) + 2s_0}{\lambda_0 - \varepsilon} = \frac{(\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_0}{\lambda_0 - \varepsilon} \]

\[ z_1^{(0)} z_2^{(0)} = \frac{2q(\mu - \varepsilon)}{\lambda_0 - \varepsilon} \]

and hence

\[ z_1^{(0)} + z_2^{(0)} = \frac{(\lambda_0 - \varepsilon) + 2q(\mu - \varepsilon) + s_0}{\lambda_0 - \varepsilon} \]

\[ z_1^{(0)} z_2^{(0)} = \frac{2q(\mu - \varepsilon)}{\lambda_0 - \varepsilon} \]

\[ \bar{P}_0(s_0) = \frac{z_1^{(0)^2} + 2q(\mu - \varepsilon) z_1^{(0)^{r+1}} \bar{P}_{r+1}(s_1)}{s_0 z_1^{(0)}} \]

... (4.6) Using Rouche’s theorem in (4.4) we get...

Using the result (4.7) in (4.4) we get

\[ z^2 - (1-z)[2q(\mu - \varepsilon) - (\lambda_0 - \varepsilon)z] \]

\[ \bar{P}(z,s_0) = \frac{z_1^{(0)^2} + 2q(\mu - \varepsilon) z_1^{(0)^{r+1}} \bar{P}_{r+1}(s_1)}{s_0 z_1^{(0)}} \]

... (4.8)

\[ \bar{P}_0(s_0) = \bar{P}(0,s_0) = \frac{-2q(\mu - \varepsilon)}{-(\lambda_0 - \varepsilon)(z_1^{(0)} z_2^{(0)})} \]

\[ = \frac{2q(\mu - \varepsilon)}{s_0 (\lambda_0 - \varepsilon) z_2^{(0)}} + \frac{2q(\mu - \varepsilon)}{s_0 z_2^{(0)^r}} \left( \frac{2q(\mu - \varepsilon)}{\lambda_0 - \varepsilon} \right)^r \bar{P}_{r+1}(s_1) \]

\[ = \frac{2q(\mu - \varepsilon)}{s_0 (\lambda_0 - \varepsilon)} \left[ 1 + \frac{(\lambda_0 - \varepsilon) z_2^{(0)^r}}{z_2^{(0)^r}} \bar{P}_{r+1}(s_1) \right] \]

\[ = \frac{2q(\mu - \varepsilon)}{(\lambda_0 - \varepsilon) z_2^{(0)}} + 2q(\mu - \varepsilon) \left[ \frac{2q(\mu - \varepsilon)}{\lambda_0 - \varepsilon} \right]^r \frac{\bar{P}_{r+1}(s_1)}{z_2^{(0)^r}} \]

... (4.9)

From equation (4.9) we get

\[ \frac{d}{ds_0} \left[ s_0 \bar{P}_0(s_0) \right] = \frac{d}{ds_0} \left[ \frac{2q(\mu - \varepsilon)}{(\lambda_0 - \varepsilon) z_2^{(0)}} + 2q(\mu - \varepsilon) \left[ \frac{2q(\mu - \varepsilon)}{\lambda_0 - \varepsilon} \right]^r \frac{\bar{P}_{r+1}(s_1)}{z_2^{(0)^r}} \right] \]

\[ = \frac{d}{ds_0} \left[ s_0 \bar{P}_0(s_0) \right] \]

... (4.9)
From equations (4.10) and (4.11) we get

\[
E\{T_{\text{busy}}^{(0)}\} = \frac{1}{1 - \rho(0)} \left[ \frac{1}{\lambda_0 - \varepsilon} + \left( \frac{2}{\rho(0)} \right)^{r} r \frac{L_t}{s_{\text{busy}}} \overline{P}_{r+1}(s_{1}) \right] \]

\[
: \text{The average length of the busy period when the system is in faster rate of arrivals either with feedback or without feedback is given by}
\]

\[
E\{r_{\text{busy}}^{(0)}\} = \frac{1}{1 - \rho(0)} \left[ \frac{1}{\lambda_0 - \varepsilon} + \left( \frac{2}{\rho(0)} \right)^{r} r \frac{L_t}{s_{\text{busy}}} \overline{P}_{r+1}(s_{1}) \right] \]

which is consistent with the corresponding result of the conventional model discussed by Gross and Harris (1974) for the case \( \varepsilon = 0, p=q=1, \lambda_0 = \lambda_1 = \lambda \) and is also consistent with the corresponding result of the busy period analysis of \( M/M/1/\infty \) model discussed by Thiagarajan and Srinivasan for the case \( p=q=1 \).

From (3.6) to (3.10) are multiplied through by \( z^{n} \), \( n = r+1, r+2, r+3, ..., R-1, R, R+1, ... \), and summing on \( n \) from \( n = r+1, r+2, ..., R-1, R+1, ... \), it is found that

\[
\frac{d}{dt_{1}} \left[ P(z, t_{1}) \right] = \sum_{n=r+1}^{\infty} P_{n}(t_{1}) z^{n} \]

Using the procedure as in the case when the system is in faster rate of arrivals either with feedback or without feedback we get

\[
\frac{d}{dt_{1}} \left[ P(z, t_{1}) \right] = P_{r+1}^{'}(t_{1}) z^{r+1} + P_{r+2}^{'}(t_{1}) z^{r+2} + \ldots + P_{R-1}^{'}(t_{1}) z^{R-1}
\]

\[+
\]

\[+
\]

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\]

\[+
\]
Using Rouehe’s theorem in (4.14) we get

\[
\begin{align*}
\frac{d}{dt} P(z,t_1) &= \left[ (\lambda_1 - \varepsilon) z^2 - \left( (\lambda_1 - \varepsilon) + 2q(\mu - \varepsilon) \right) z + 2q(\mu - \varepsilon) \right] \sum_{n=0}^{\infty} P_n(t_1) z^n \\
&\quad + \left[ \left( (\lambda_1 - \varepsilon) + 2q(\mu - \varepsilon) \right) \right] P_{z+1}(1) z^{r+1} \\
&\quad + (\lambda_0 - \varepsilon) R_{z-1}(0) z^{R-1}
\end{align*}
\]

Taking Laplace Transform on both side we get

\[
\begin{align*}
z \left[ s \tilde{P}(z,s_1) - P(z,r+1) \right] &= \left[ (\lambda_1 - \varepsilon) z^2 - \left( (\lambda_1 - \varepsilon) + 2q(\mu - \varepsilon) \right) z + 2q(\mu - \varepsilon) \right] \tilde{P}(z,s_1) \\
&\quad + \left[ (\lambda_1 - \varepsilon) z^2 - \left( (\lambda_1 - \varepsilon) + 2q(\mu - \varepsilon) \right) z + 2q(\mu - \varepsilon) \right] \\
&\quad - (\lambda_0 - \varepsilon) R_{z-1}(s_0) z^{R+1}
\end{align*}
\]

\[
\begin{align*}
\tilde{P}(z,s_1) &= \frac{z^{r+2} - \left( (\lambda_1 - \varepsilon) z^2 - \left( (\lambda_1 - \varepsilon) + 2q(\mu - \varepsilon) + s_1 \right) z + 2q(\mu - \varepsilon) \right) \tilde{P}(z,s_1) + (\lambda_0 - \varepsilon) R_{z-1}(s_0) z^{R+1}}{\left( (\lambda_1 - \varepsilon) + 2q(\mu - \varepsilon) + s_1 \right) z - (\lambda_1 - \varepsilon) z^2 - 2q(\mu - \varepsilon)}
\end{align*}
\]

where

\[
L \{ P(z,t_1) = \tilde{P}(z,s_1), L \{ P_{z+1}(t_1) = \tilde{P}_{z+1}(s_1) \} \text{ and } P(z,r+1) = z^{r+1}
\]

Since the Laplace Transform \( \tilde{P}(z,s_1) \) Converges in the region \( |z| \leq 1 \), \( \text{Re} (s_1) \geq 0 \), wherever the denominator of the right hand side of (4.14) has zeros in that region, so must the numerator, the denominator has two zeros.

Using Rouehe’s theorem in (4.14) we get

\[
\begin{align*}
\tilde{P}_{z+1}(s_1) &= \frac{z^{r+1} - (\lambda_0 - \varepsilon) R_{z-1}(s_0) z^{R+1}}{z_1^{(i+1)}} \\
\tilde{P}(z,s_1) &= \frac{z^{r+2} - (1-z) [2q(\mu - \varepsilon) - (\lambda_1 - \varepsilon)] z^{r+1} \tilde{P}_{z+1}(s_1) + (\lambda_0 - \varepsilon) z^{R+1} R_{z-1}(s_0)}{(\lambda_1 - \varepsilon) (z - z_1^{(i)}) (z_2^{(i+1)} - z)}
\end{align*}
\]
where

\[
z_1^{(i)} = \left[ (\lambda_1 - \varepsilon) + 2q(\mu - \varepsilon) + s_1 \right] - \sqrt{\left[ (\lambda_1 - \varepsilon) + 2q(\mu - \varepsilon) + s_1 \right]^2 - 8(\lambda_1 - \varepsilon)q(\mu - \varepsilon)} \frac{1}{2(\lambda_1 - \varepsilon)}
\]

\[
z_2^{(i)} = \left[ (\lambda_1 - \varepsilon) + 2q(\mu - \varepsilon) + s_1 \right] + \sqrt{\left[ (\lambda_1 - \varepsilon) + 2q(\mu - \varepsilon) + s_1 \right]^2 - 8(\lambda_1 - \varepsilon)q(\mu - \varepsilon)} \frac{1}{2(\lambda_1 - \varepsilon)}
\]

... (4.17)

\[L[P(z,t_1)] = \bar{P}(z,s_1), L[P_{R^{-1}}(t_0)] = \bar{P}_{R^{-1}}(s_0)\]

It is clear that \[|z_1^{(i)}| < |z_2^{(i)}|\] from equation (4.17)

\[
z_1^{(i)} + z_2^{(i)} = \frac{(\lambda_1 - \varepsilon) + 2q(\mu - \varepsilon) + s_1}{(\lambda_1 - \varepsilon)}
\]

\[
z_1^{(i)}, z_2^{(i)} = \frac{2q(\mu - \varepsilon)}{(\lambda_1 - \varepsilon)}
\]

... (4.18)

Using (4.15) in (4.16) we get

\[
z_1^{(i+1)} - (1 - z_1^{(i)}) \left[ 2q(\mu - \varepsilon) - (\lambda_1 - \varepsilon)z_1^{(i)} \right] z_1^{(i+1)} \bar{P}_{R+1}(s_1) + (\lambda_0 - \varepsilon) \bar{P}_{R^{-1}}(s_0) z_1^{(i+1)} = 0
\]

\[
\bar{P}_{R+1}(s_1) = \frac{z_1^{(i+1)} + (\lambda_0 - \varepsilon) \bar{P}_{R^{-1}}(s_0) z_1^{(i+1)}}{(1 - z_1^{(i)}) \left[ 2q(\mu - \varepsilon) - (\lambda_1 - \varepsilon)z_1^{(i)} \right] z_1^{(i+1)}} \]

\[
\bar{P}_{R+1}(s_1) = \frac{z_1^{(i+1)} + (\lambda_0 - \varepsilon) \bar{P}_{R^{-1}}(s_0) z_1^{(i+1)}}{(1 - z_1^{(i)}) \left[ 2q(\mu - \varepsilon) - (\lambda_1 - \varepsilon)z_1^{(i)} \right] z_1^{(i+1)}} ... (4.19)
\]

From equations (4.15), (4.17), (4.18) and (4.19) we get

\[
P(z,s_1) = \frac{(\lambda_1 - \varepsilon)(z - z_1^{(i)})}{s_1(z_1^{(i)})} \left[ z_1^{(i+1)} + (\lambda_0 - \varepsilon) \bar{P}_{R^{-1}}(s_0) z_1^{(i+1)} \right] z_1^{(i+1)}
\]

\[
P(z,s_1) = \frac{(\lambda_1 - \varepsilon)(z - z_1^{(i)})}{s_1(z_1^{(i)})} \left[ z_1^{(i+1)} + (\lambda_0 - \varepsilon) \bar{P}_{R^{-1}}(s_0) z_1^{(i+1)} \right] z_1^{(i+1)}
\]

\[
P(z,s_1) = \frac{(\lambda_1 - \varepsilon)(z - z_1^{(i)})}{s_1(z_1^{(i)})} \left[ z_1^{(i+1)} + (\lambda_0 - \varepsilon) \bar{P}_{R^{-1}}(s_0) z_1^{(i+1)} \right] z_1^{(i+1)}
\]

\[
\bar{P}(z,s_1) = \frac{(\lambda_1 - \varepsilon)(z - z_1^{(i)})}{s_1(z_1^{(i)})} \left[ z_1^{(i+1)} + (\lambda_0 - \varepsilon) \bar{P}_{R^{-1}}(s_0) z_1^{(i+1)} \right] z_1^{(i+1)}
\]

\[
\bar{P}(z,s_1) = \frac{(\lambda_1 - \varepsilon)(z - z_1^{(i)})}{s_1(z_1^{(i)})} \left[ z_1^{(i+1)} + (\lambda_0 - \varepsilon) \bar{P}_{R^{-1}}(s_0) z_1^{(i+1)} \right] z_1^{(i+1)}
\]

\[
\bar{P}(z,s_1) = \frac{(\lambda_1 - \varepsilon)(z - z_1^{(i)})}{s_1(z_1^{(i)})} \left[ z_1^{(i+1)} + (\lambda_0 - \varepsilon) \bar{P}_{R^{-1}}(s_0) z_1^{(i+1)} \right] z_1^{(i+1)}
\]
\[
\overline{P}(0,s_1) = -\frac{2q(\mu-\epsilon)}{\lambda_1-\epsilon} \left[ z_1^{(1)} + (\lambda_0-\epsilon)P_{R-1}(s_0)z_1^{(1)} \right] \]

\[
\overline{P}(0,s_1) = \frac{z_1^{(1)} + (\lambda_0-\epsilon)\overline{P}_{R-1}(s_0)z_1^{(1)}}{s_1} \quad \ldots (4.20)
\]

\[
s_1\overline{P}(0,s_1) = z_1^{(1)} + (\lambda_0-\epsilon)\overline{P}_{R-1}(s_0)z_1^{(1)}
\]

\[
= \left[ \frac{1}{z_2^{(1)}} \frac{2q(\mu-\epsilon)}{\lambda_1-\epsilon} \right]^{R+1} + (\lambda_0-\epsilon)\overline{P}_{R-1}(s_0) \left[ \frac{2q(\mu-\epsilon)}{z_2^{(1)}} \right]^{-R}
\]

\[
s_1\overline{P}(0,s_1) = \left[ \frac{1}{z_2^{(1)}} \frac{2q(\mu-\epsilon)}{\lambda_1-\epsilon} \right]^{R+1} + (\lambda_0-\epsilon)\overline{P}_{R-1}(s_0) \left[ \frac{2q(\mu-\epsilon)}{\lambda_1-\epsilon} \right]^{-R}
\]

\[
\frac{d}{ds_1} \left[ s_1\overline{P}(0,s_1) \right]_{s_1=0} = \frac{1}{(\lambda_1-\epsilon) - 2q(\mu-\epsilon)} \left[ (r+1) \left( \frac{2q(\mu-\epsilon)}{\lambda_1-\epsilon} \right)^{R+1} \right] = -Lt_{s_1=0} (\lambda_0-\epsilon)\overline{P}_{R-1}(s_0)R \left[ \frac{2q(\mu-\epsilon)}{\lambda_1-\epsilon} \right]^{-R}
\]

\[
-\frac{d}{ds_1} \left[ s_1\overline{P}(0,s_1) \right]_{s_1=0} = \frac{1}{(\lambda_1-\epsilon) - 2q(\mu-\epsilon)} \left[ (r+1) \left( \frac{2q(\mu-\epsilon)}{\lambda_1-\epsilon} \right)^{R+1} \right] = - Lt_{s_1=0} (\lambda_0-\epsilon)\overline{P}_{R-1}(s_0)R \left[ \frac{2q(\mu-\epsilon)}{\lambda_1-\epsilon} \right]^{-R}
\]

\[
\frac{d}{ds_1} \left[ s_1\overline{P}(1,s_1) \right]_{s_1=0} = \frac{d}{ds_1} \left[ \int_0^\infty e^{-s_1t}P_{r+1}(t_1)dt_1 \right]_{s_1=0} = \left[ \int_0^\infty e^{-s_1t}t_1P_{r+1}(t_1)dt_1 \right]_{s_1=0} = -\int_0^\infty t_1d(P_{r+1}(t)) = -E \left\{ T^{(1)}_{bus} \right\}
\]

\[
-\frac{d}{ds_1} \left[ s_1\overline{P}(1,s_1) \right]_{s_1=0} = E \left\{ T^{(1)}_{bus} \right\} \quad \ldots (4.21)
\]

From (4.21) and (4.22) we get

\[
E \left\{ T^{(1)}_{bus} \right\} = \frac{1}{(\lambda_1-\epsilon) - 2q(\mu-\epsilon)} \left[ (r+1) \left( \frac{2q(\mu-\epsilon)}{\lambda_1-\epsilon} \right)^{R+1} \right] = - Lt_{s_1=0} (\lambda_0-\epsilon)\overline{P}_{R-1}(s_0)R \left[ \frac{2q(\mu-\epsilon)}{\lambda_1-\epsilon} \right]^{-R}
\]

and hence
The average length of the busy period when the system is in the slower rate of arrivals either with feedback or without feedback, is

\[ E\{T_{busy}^{(1)}\} = \frac{1}{2q(\mu - \varepsilon)[\rho(1) - 1]} \left[ r + \frac{2}{\rho(1)} \right]^{r+1} + \text{Lt}_{s_i \rightarrow 0} \left( \lambda_{0} - \varepsilon \right) \overline{P}_{R-1} \left( s_{0} \right) R \left[ \frac{2}{\rho(1)} \right]^{R} \]

... (4.23)

which is consistent with the corresponding result of the conventional model discussed by Gross and Harries (1974) for case \( \varepsilon = 0, \lambda_{0} = \lambda, p = q = 1, \) the present model reduces to busy period analysis of M/M/1/\( \infty \) interdependent model with controllable arrival rates discussed by Thiagarajan and Srinivasan [6] for the case \( p=q=1, \) Two server = single server (2q=\( q \mu = q \mu \)).

5. Numerical Illustrations

For various values of \( r, R, \lambda_{0}, \lambda_{1}, \mu, \) and \( \varepsilon, \) \( E\{T_{busy}^{(0)}\} \) and \( E\{T_{busy}^{(1)}\} \) are computed and tabulated by taking \( p=q=\frac{1}{2}. \)

<table>
<thead>
<tr>
<th>R</th>
<th>R</th>
<th>( \lambda_{0} )</th>
<th>( \lambda_{1} )</th>
<th>( \mu )</th>
<th>( \varepsilon )</th>
<th>( E{T_{busy}^{(0)}} )</th>
<th>( E{T_{busy}^{(1)}} )</th>
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<td>10</td>
<td>5</td>
<td>4</td>
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<td>1.888</td>
<td>164.10</td>
</tr>
<tr>
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<tr>
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<tr>
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<td>1.866</td>
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</tbody>
</table>

i. It is observed from the Table 5.1 that the average length of the busy period is increase when the system is either in faster or in slower rate of arrivals by increasing the value of \( r \) while the other parameters were fixed.

ii. The average length of the busy period rapidly increases when the system is in faster rate of arrivals by increasing the value of \( \lambda_{0} \) and keeping the other parameters were fixed.

iii. The average length of busy period rapidly decreases by increasing the value of \( \mu \) and keeping the other parameters were fixed while the system is in faster rate of arrivals.

iv. The average length of busy period increases by increasing the value of \( \mu \) and keeping the other parameters were fixed while the system is in slower rate of arrivals.

v. The average length of busy period increases while the system is in faster rate of arrivals by increasing the value of \( \varepsilon \) keeping the other parameters were fixed.

vi. The average length of busy period increases while the system is in slower rate of arrivals by increasing the value of \( \varepsilon \) and keeping the other parameters were fixed.

vii. The average length of busy period decreases while the system is in slower rate arrival by decreasing the value of \( R \) keeping the other parameters were fixed.

viii. The average length of busy period is remain unchanged while the system is in faster rate of arrivals by increasing the value of \( R \) keeping the other parameters were fixed.

References


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