International Journal of Application or Innovation in Engineering & Management (IJAIEM)
Web Site: www.ijaiem.org Email: editor@ijaiem.org
Volume 4, Issue 9, September 2015 ISSN 2319 - 4847

Power series Method for solving first order stiff systems on Piecewise uniform mesh

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ABSTRAC

In this paper, a power series method is suggested on a piecewise uniform mesh to solve a system of initial value problems of first order stiff Ordinary Differential Equations(ODEs). The numerical results exhibited the performance of this method and the results revealed that this method is very effective and convenient than the power series method without piecewise uniform mesh.

Key words: System of stiff differential equations, Initial value problem, Power series method, Piecewise uniform mesh.

1 INTRODUCTION

The study of differential equations in pure and applied mathematics, physics and engineering is extensive. All of these disciplines are concerned with the properties of differential equations of various types. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modelling virtually every physical, technical, or biological process from celestial motion, to bridge design, to interactions between neurons. Differential equations like those used to solve real - life problems may not essentially be directly solvable, i.e. need not have closed form solutions instead solutions can be approximated using numerical methods. Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to perfect the behavior of complex systems. The mathematical theory of differential equations first developed together with the sciences, where the equations had originated and where the results initiated application. However, diverse problems, sometimes originating in quite distinct scientific fields, may give rise to identical differential equations. Whenever this happens, mathematical theory behind the equations can be viewed as a unifying principle behind diverse phenomena. For example, consider propagation of light and sound in the atmosphere, and of waves on the surface of a pond.

In the past power series method played a minor role in the numerical solutions of ordinary and partial differential equations as it is often difficult to operate with power series. However, power series have certain advantages. A truncated series forms a closed approximation of the solution which can be evaluated at any point in the region where the series converges. Instability, which causes difficulties for finite difference solutions, does not affect the power series solutions. The series solution has a great accuracy which permits the study of analytic properties of the solution to an extent which is unachievable with a finite difference solution, the series solution can be used as an intermediate result which can be integrated and differentiated easily. If a finite difference solution is only an intermediate step in the solution of a problem, computer storage problems can be a major concern.

In [22], Tomasz Blenski states that, in a system of equations the time constant characterizing the system differ from each other by orders of magnitude. Such a system of ODEs is called as stiff system of equations. For this stiff systems, in initial moment (close to origin) the standard perturbation approach breaks down for t. Therefore for small t the standard expansion be supplemented by an inner solution. This approach of stiff system of ODEs leads to the singular perturbation method. Singular perturbation problem (spp) works well in piecewise uniform mesh rather than uniform mesh which is discussed in [14]. So in stiff system also we can implemented the piecewise uniform mesh logic.

For a detailed discussion on ODEs, stiff nature, stiff ODEs and power series method one may refer to [1, 2, 3, 7, 8, 9, 11, 13, 21] and to name a few.

In this paper, the Power series method proposed by Nuran Guzel, Mustafa Bayram [15] for the numerical solution of initial value problems in ODEs are reviewed. The focus of this paper is to improve the performance of the Power series method by applying it in a piecewise uniform mesh (like Shishkin mesh). One can refer [17], [18], [19] and [20] for numerical methods for stiff ODEs on piecewise uniform mesh. Possible approaches to the construction of such piecewise uniform meshes and also some special schemes are given in [6, 12, 14]. As a result, the proposed method have improved the accuracy and required less computational time.

Numerical solution of dynamic systems is often leads to stiff systems of first order ODEs with Initial Value Problems (IVPs) in general form,

$$y_i^{'} = f_i(t, y_i) \qquad (1)$$

with initial condition

$$y_i(t_0) = y_{0,i}, \qquad i = 1, 2, ..., n \quad t \in [0, T]$$

As given in [15], the solutions of (1) can be assumed that

$$y = y_0 + et, \qquad (2)$$

where e is a vector function which is the same size as y_0. Substitute (2) into (1) and neglect higher order term. We have the linear equation of e in the form

$$Ae = B, \qquad (3)$$

where A and B is a constant matrixes. Solve this equation of (3), the coefficients of t in (2) can be determined. Repeating above procedure for higher order terms, we can get the arbitrary order power series of the solutions for (1).

2 DESCRIPTION OF THE METHOD

As given in [15], we define another type power series in the form

$$f(x) = f_0 + f_1 t + f_2 t^2 + ... + (f_n + p_1 e_1 + ... + p_m e_m) t^n, \qquad (4)$$

where $p_1, p_2, ..., p_m$ are constant. $e_1, e_2, ..., e_m$ are basis of vector e, m is size of vector e, y is a vector with m elements in (2). Every element can be represented by the power series in (4).

$$y_i = y_{i,0} + y_{i,1} t + y_{i,2} t^2 + ... + e_i t^n, \qquad (5)$$

where y_i is i^{th} element of y. Substitute (5) into (1), we can get the following

$$f_i = (f_{i,n} + p_{i,1} e_1 + ... + p_{i,m} e_m) t^{n-j} + Q(t^{n-j+1}) \qquad (6)$$

where f_i is i^{th} element of $f(y, y^{'}, t)$ in (1) and j is 0 if $f(y, y^{'}, t)$ have $y^{'}, 1$ if do not. From (6) and (3), we can determine the linear equation in (3) as follow

$$A_{i,j} = P_{i,j}, \qquad (7)$$

$$B_i = -f_{i,n}. \qquad (8)$$

Solve this linear equation, we have e_i $(i = 1, ..., m)$. Substitute e_i into (5), we have $y_i (i = 1, ..., m)$ which are polynomials of degree n. Repeating this procedure from (5) to (8), we can get the arbitrary order power series of the solution for differential-algebraic equations in (1).

3 DESCRIPTION OF PIECEWISE UNIFORM MESH

A piecewise uniform mesh is constructed on the interval [0,T] as follows:

Choose a point σ satisfying $0 < \sigma \le \frac{T}{4}$ and assume that $N = 2^m$ with $m \ge 2$. The point σ is called a transition point and it divides the interval [0,T] into the two subintervals:

$$[0, T] = [0, \sigma] \cup [\sigma, T].$$

If the solution has fast varying component in the neighbourhood of $t = 0$, it is natural to have more number of mesh points in the neighbourhood of $t = 0$. Therefore the corresponding piecewise uniform mesh is constructed by dividing $(0, \sigma)$ into $\frac{N}{4}$ equal mesh elements and (σ, T) into $\frac{3N}{4}$ equal mesh elements. This will give a better information about the solution near $t = 0$.

The piecewise uniform mesh is used with the following location of the transition point

$$\sigma = \mathbf{min}\{\frac{T}{4}, \varepsilon \ln N\} \qquad (9)$$

Consider the parameter ε as

$$\varepsilon = 10^{(-6)} \qquad (10)$$

and

$$\begin{cases} t_j = jh_1 \quad \text{where} \quad h_1 = \dfrac{4\sigma}{N}, \quad j = 0(1)\dfrac{N}{4}, \\ t_j = \sigma + (j - \dfrac{N}{4})h_2 \quad \text{where} \quad h_2 = \dfrac{4(T-\sigma)}{3N}, \quad j = (\dfrac{N}{4}+1)(1)N. \end{cases} \qquad (11)$$

If $\sigma = T/4$, (i.e)., $\dfrac{T}{4} < \varepsilon \ln N$ then $h = N^{-1}$.

In such a case the method can be analysed using the standard techniques. We therefore assume that

$$\sigma = \varepsilon \ln N \qquad (12)$$

Let step size of t to be h and substitute it into the power series of y and y', we have y and y' at t. If we repeat above procedure, we have numerical solution of differential-algebraic equations in (1) [10, 16]. This method has been used to obtained approximate numerical and theoretical solutions of a large class of differential-algebraic equation [4, 5] references therein.

4 NUMERICAL EXAMPLE

The numerical results of power series method with piecewise uniform mesh is compared with uniform mesh. The comparison is based in terms of maximum error and average error. These results are tabulated and the numerical results are recorded in terms of the following quantities:

As the formula given in [17], [18] and [19] for uniform mesh we have,

$h = \dfrac{(b-a)}{N}$, where b is the end value of t and a is the initial value of t.

The calculation of error (for all piecewise uniform mesh and uniform mesh) is given as:

$$\text{error}_j = | u_{j(\text{exactsolution})} - u_{j(\text{approximate})} |.$$

For maximum error(MAXE)(for all piecewise uniform mesh and uniform mesh), we compute using the formula which is defined as follows:

$$\text{MAXE}^N = \mathbf{max}(\text{error}_j)$$

The average error for power series method with piecewise uniform mesh is defined as:

$$\text{AVE1} = \frac{\sum_{j=1}^{\frac{N}{4}} (\text{error}_j)}{N_1} \quad \text{where } N_1 = \frac{4\sigma}{h_1}, \qquad j = 0(1)\frac{N}{4},$$

$$\text{AVE2} = \frac{\sum_{j=\frac{N}{4}+1}^{N} (\text{error}_j)}{(N_2)} \quad \text{where } N_2 = \frac{4(T-\sigma)}{3h_2}, \quad j = (\frac{N}{4}+1)(1)N.$$

$$\text{AVE} = \max\{\text{AVE1}, \text{AVE2}\}$$

The average error for power series method with uniform mesh is defined as:

$$\text{AVE} = \begin{cases} \dfrac{\Sigma_{j=1}^{N} \text{error}_j}{N} \quad \text{where} N = \dfrac{(b-a)}{h}, \end{cases}$$

b is the end value of t and a is the initial value of t.

Example 4.1 An illustrating example of a stiff system given in [15], be discussed now.

$$u'(t) = -1002u(t) + 1000v^2(t) \qquad (13)$$

$$v'(t) = u(t) - v(t)(1 + v(t)) \quad \forall t \in [0,1], \qquad (14)$$

$$u(0) = 1, v(0) = 1. \qquad (15)$$

Exact solution:

$$u(t) = e^{-2t}$$

$$v(t) = e^{-t}.$$

If we apply the power series method [15] to the given equation of system, the solutions of (4.1) and (4.2) can be supposed as

$$u(t) = u + e_1 t \Rightarrow u(t) = 1 + e_1 t \qquad (16)$$

$$v(t) = v + e_2 t \Rightarrow v(t) = 1 + e_2 t \qquad (17)$$

Substitude (4.4) and (4.5) into (4.1) and (4.2), and neglect higher order terms, we have

$$e_1 + 2 + Q(t) = 0 \qquad (18)$$

$$1 + e_2 + Q(t) = 0. \qquad (19)$$

These formulas correspond to (2.3). The linear equation that corresponds to (2.4) can be given in the following

$$Ae = B, \qquad (20)$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} -2 \\ -1 \end{bmatrix},\ e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

From equation (4.8), we have linear equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Solve this linear equation, we have

$$e = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

and

$$u(t) = 1 - 2t, \qquad (21)$$

$$v(t) = 1 - t \qquad (22)$$

From (4.9) and (4.10) the solutions of (4.1) and (4.2) can be supposed as

$$u(t) = 1 - 2t + e_1 t^2, \qquad (23)$$

$$v(t) = 1 - t + e_2 t^2 \qquad (24)$$

In this manner, substitute (4.11) and (4.12) into (4.1) and (4.2) and neglect higher order terms, we have

$$(2e_1 - 4)t + Q(t^2) = 0, \quad (25)$$

$$(2e_2 - 1)t + Q(t^2) = 0 \quad (26)$$

where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2, \end{bmatrix}\ B = \begin{bmatrix} 4 \\ 1, \end{bmatrix}\ e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

From (4.13) and (4.14) we have linear equation

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Solve this linear equation,we have

$$e = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$$

Therefore

$$u(t) = 1 - 2t + 2t^2, \qquad (27)$$

$$v(t) = 1 - t + \frac{1}{2}t^2. \qquad (28)$$

Repeating above procedure we have

$$u(t) = 1 - 2t + 2t^2 - \frac{4}{3}t^3 + \frac{2}{3}t^4...., \qquad (29)$$

$$v(t) = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4..... \qquad (30)$$

Further proceed with the equations (29) and (30) by applying the piecewise uniform mesh method. The piecewise uniform mesh is used with the following location of the transition point

$$\sigma = \mathbf{min}\{\frac{T}{4}, \varepsilon \ln N\} \qquad (31)$$

where ε as

$$\varepsilon = 10^{(-6)} \qquad (32)$$

and

$$\begin{cases} t_j = jh_1 \quad \text{where} \quad h_1 = \frac{4\sigma}{N}, \quad j = 0(1)\frac{N}{4}, \\ t_j = \sigma + (j - \frac{N}{4})h_2 \quad \text{where} \quad h_2 = \frac{4(T-\sigma)}{3N}, \quad j = (\frac{N}{4}+1)(1)N \\ \text{where } N = 2^m, m \geq 2. \end{cases} \qquad (33)$$

And the numerical results obtained by applying the piecewise uniform mesh method to the example 4.1 are given in table 1.

Example 4.2 Let us consider the following system of differential equation discussed in [15], be done with piecewise uniform mesh.

$$u'(t) = -20u - 0.25v - 19.75w, \quad u(0) = 1,$$
$$v'(t) = 20u - 20.25v + 0.25w, \quad v(0) = 0,$$
$$w'(t) = 20u - 19.75v - 0.25w, \quad w(0) = -1,$$

The analytic solution of the problem is

$$u = [\exp^{-\frac{1}{2t}} + \exp^{-20t}(\cos(20t) + \sin(20t))]/2,$$
$$v = [\exp^{-\frac{1}{2t}} - \exp^{-20t}(\cos(20t) - \sin(20t))]/2,$$
$$w = -[\exp^{-\frac{1}{2t}} + \exp^{-20t}(\cos(20t) - \sin(20t))]/2.$$

If we apply the power series method [15] to the given equation system, following solution is obtained,

$$u = 1 - \frac{1}{4}t - \frac{3199}{16}t^2 + \frac{85333}{32}t^3 - \frac{3413333}{256}t^4 - \frac{33}{250000}t^5 + \frac{12799999}{36}t^6...., \qquad (34)$$

$$v = \frac{79}{4}t - \frac{3199}{16}t^2 - \frac{1}{96}t^3 + \frac{3386667}{256}t^4 - 106666t^5 + \frac{1262212226}{355}t^6...., \qquad (35)$$

$$w = -1 + \frac{81}{4}t - \frac{3201}{16}t^2 + \frac{1}{96}t^3 + \frac{3413333}{256}t^4 - \frac{213333333}{2000}t^5 + \frac{21688755}{61}t^6.... \qquad (36)$$

Further proceed the Power series method by applying the piecewise uniform mesh method (9), (10) and (11) to (34), (35) and (36). And the numerical results obtained by applying the piecewise uniform mesh method to the example 4.2 are given in table 2.

5 CONCLUSION

The method has been applied to the solution of stiff systems of differential equation with Piecewise uniform mesh outperformed the power series method with uniform mesh in terms of error, average error and accuracy. Hence, the power series method with Piecewise uniform mesh is more efficient than the power series method with uniform mesh for stiff systems.

Acknowledgements The authors are grateful to Dr. N. Ramanujam, Professor of Emeritus, Bharathidasan University, Tiruchirappalli - 24 for his suggestions on this work.

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Table 1: Values of $MAXE(u), AVE(u), MAXE(v), AVE(v)$ for the solution components u and v respectively for the Example 4.1

		Number of mesh points N		
		8	16	32

MAXE(u)	piecewise uniform mesh	6.2102e-007	1.3328e-006	1.8753e-006
	uniform mesh	1.3288e-006	1.8724e-006	2.2038e-006
AVE(u)	piecewise uniform mesh	7.7628e-008	8.3298e-008	5.8603e-008
	uniform mesh	1.6611e-007	1.1703e-007	6.8867e-008
MAXE(v)	piecewise uniform mesh	1.9648e-008	4.2252e-008	5.9511e-008
	uniform mesh	4.2127e-008	5.9419e-008	6.9969e-008
AVE(v)	piecewise uniform mesh	2.456e-009	2.6407e-009	1.8597e-009
	uniform mesh	5.2658e-009	3.7137e-009	2.1865e-009

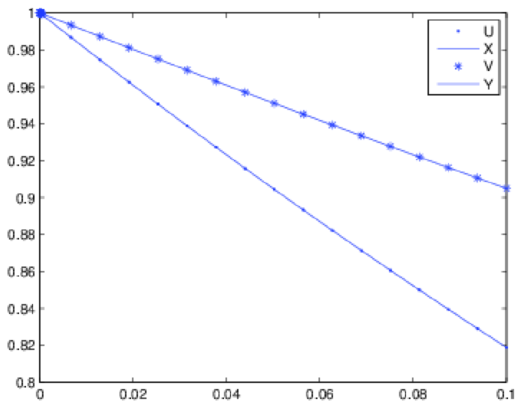

Figure 1:

For the example-1 with $n = 32, \varepsilon = 10^{(-6)}$, the solution obtained by the suggested numerical method is displayed in Figure 1.

Table 2: Values of $MAXE(u), AVE(u), MAXE(v), AVE(v)$ for the solution components u and v respectively for the Example 4.2

		Number of mesh points N		
		8	16	32
MAXE(u)	piecewise uniform mesh	0.22965e-001	0.64219e-001	0.10154 e+000
	uniform mesh	0.63190e-001	0.10069e+000	0.12560e+000
AVE(u)	piecewise uniform mesh	0.28706e-002	0.40137e-002	0.31731e-002
	uniform mesh	0.78995e-002	0.62934e-002	0.39249e-002
MAXE(v)	piecewise uniform mesh	0.41502e-003	0.58649e-002	0.12059e-001
	uniform mesh	0.57076e-002	0.11910e-001	0.16414e-001
AVE(v)	piecewise uniform mesh	0.51878e-004	0.36656e-003	0.37684e-003
	uniform mesh	0.71345e-003	0.74439e-003	0.51292e-003

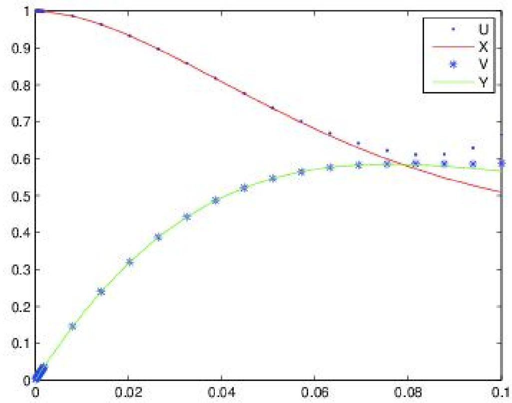

Figure 2:

For the example-2 with $n = 32, \varepsilon = 10^{(-6)}$, the solution obtained by the suggested numerical method is displayed in Figure 2.