Multipole Moment Expansion of Einstein-Maxwell Expansion

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ABSTRACT

We study multipole expansion for stationary axisymmetric Einstein-Maxwell (EM) equations. In this analysis we mainly follow the work of Forrester [8] who formulated a multipole expansion scheme for pure Einstein stationary axisymmetric equations following the work of Janis and Newman [7], Lamb [10], Geroch [4] and Hansen [5]. Forrester used the coordinate system used by Boyer and Lindquist [9], which turns out to be suitable for this purpose.

Keywords: about four key words separated by commas

1. INTRODUCTION

Multipole moment expansion is a well known technique through which the behavior of any field far from the source of the field can be studied. This technique has been applied extensively to the electromagnetic field (P-96, Landau and Lifshitz [1], Jackson [2], P-744) and also to the linearized gravitational field in General Relativity (Sachs and Bergmann [3], Geroch [4] and Hansen [5] proposed definition for static and stationary space-time respectively. Sciam andCnrke [6] also considered multipole expansion for static space-times. Janis and Newmann [7] have considered multipole moment for axisymmetric gravitational field. Forrester [8] attempted to incorporate elements from earlier work on multipole expansions in his consideration of asymptotic expansion for the stationary axisymmetric Pure Einstein field. He used Boyer-Lindquist [9] co-ordinates which had proved fruitful in other contexts. His treatment of the problem consist essentially of the step by step integration of the axisymmetric field equations for an asymptotically flat space-time by expanding the metric in powers of a suitable radial co-ordinate $r$, the angular dependence consisting of $P_r (\cos \theta) \sin \theta P_s (\cos \theta)$. where $P_r$ and $P_s$ are Legendre functions and associated Legendre functions respectively. The prime purpose of our work is to extend Forrester’s works to the axisymmetric Einstein-Maxwell equations. Later we shall relate this work to more recent development of the subject.

2. MULTIPOLe MOMENTS FOR A NON-LINEAR MODEL

As a prelude to considering multipole moments for Einstein Maxwell equations, we consider a non-linear model. Let us first consider a massless free scale field $\phi$ with the usual Lagrangaen:

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi$$

then corresponding field equation being

$$\partial_{\mu} \partial^{\mu} \phi = \left( \frac{\partial^2}{\partial r^2} - \nabla^2 \right) \phi = 0$$

(2)

This is the flat-space wave equation.

For static field, we get the Laplace equation

$$\nabla^2 \phi = 0$$

(3)

For axisymmetric fields, this has the well known solution in spherical co-ordinates $\left( r, \theta, \phi \right)$ givenby,

$$\phi = \sum_{l=0}^{\infty} \left( \begin{array}{c} 2l+1 \\ l \end{array} \right) \left( \begin{array}{c} 2l+1 \\ l \end{array} \right) \tilde{F}_l (\cos \theta)$$

(4)

Where, $\tilde{F}_l (\cos \theta)$ is the Legendre function of order $l$. Consider now the massless scalar field $\phi$ with the following non-linear interactions,

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^4$$

(5)

We have $\frac{\partial^2 \phi}{\partial r^2} = \psi \phi$, $\frac{\partial^2 \phi}{\partial \theta^2} = -2 \psi \phi$, $\psi_{\phi r} = \psi \phi$

(6)

from where by using $\partial_r \left( \frac{\partial \phi}{\partial r} \right) - \frac{\partial \phi}{\partial r} = 0$

(7)
we get the field equation,
\[ \sigma_\mu^\nu \phi + \phi \delta^\mu_\nu = 0 \]  
(8)

If the canonically conjugate field, \( \Pi = \frac{\partial L}{\partial \phi'} = \frac{\delta \phi}{\delta t} \),

Then the Hamiltonian Density
\[ \Pi \phi' - L = \Pi \phi' - \left( \frac{1}{2} \sigma_\mu^\nu \phi \delta^\mu_\nu - \frac{1}{2} \lambda \phi^2 \right) \]  
(9)

Which is positive definite for \( \lambda > 0 \), as expected for energy density.

The static form of (8) is then given by
\[ \nabla^2 \phi - \lambda \phi = 0 \]  
(10)

For \( \lambda = 0 \), we get the Laplace equation,
\[ \nabla^2 \phi = 0, \]  
which has a solution in spherical polar coordinates as follows.
\[ \Phi = \frac{1}{r^i} F(r) \cos \theta \]  
(11)

The functions \( U(\nu) \) and \( P(\cos \theta) \) can be obtained by separating variables as follows
\[ U(\nu) = A_{\nu} \log(r) + B_{\nu} \]  
(12)
\[ Q(\nu) = \nu \log(\sec \theta) \]  
(13)

with \( P = F(r) \) satisfying Legendre’s equation.
\[ \frac{d}{dx} \left( x^2 \frac{d}{dx} F(x) \right) + \left[ x^2 + 1 - \frac{\nu^2}{x^2} \right] F(x) = 0 \]  
(14)

For \( m = 0 \), \( x = \cos \theta \), we get the standard form of Legendre equation of order \( \ell \).
\[ \frac{d}{dx} \left( (1 - x^2) \frac{d}{dx} \phi \right) + (\ell + 1) \phi = 0 \]  
(15)

and in this case we get the standard expansion
\[ \phi = \sum_{\ell = 0}^{\infty} A_{\ell} (\cos \theta)^\ell \]  
(16)

In spherical polar co-ordinates, (10) can be written as follows:
\[ \phi_{rr} + \frac{1}{r} \phi_r + \frac{\ell (\ell + 1)}{r^2} \phi_{\theta \theta} - \lambda \phi = 0 \]  
(17)

where, \( \phi_r = \frac{\partial \phi}{\partial r}, \phi_\theta = \frac{\partial \phi}{\partial \theta} \), etc. To determine the field far from the source, (which we assume to be in the neighborhood of \( r = 0 \)) we expand \( \phi \) in inverse power of \( \lambda \) as follows:
\[ \phi = \frac{A_1}{r} + \frac{A_2}{r^2} + \frac{A_3}{r^3} + \ldots \]  
(18)

where \( A_i \) are functions of \( \theta \), i.e. \( A_i = A_i(\theta), i = 1, 2, \ldots \)

Substituting (18) in (16) and equating the coefficients of various powers to zero, we get the following equations for the first three non-zero orders,
\[ h^{2i - 1} \cot \theta A_i + A_i^{(i)} - \lambda A_i = 0 \]  
(19a)
\[ h^{2i - 2} \cot \theta A_i + A_i^{(i)} - 2 \lambda A_i = 0 \]  
(19b)
\[ h^{2i - 3} \cot \theta A_i + A_i^{(i)} - 3 \lambda A_i = 0 \]  
(19c)

with \( \lambda = \frac{\mu}{\sigma^2} \), etc.

These are a system of non-linear differential equations and difficult to solve. We therefore assume \( \lambda \) to be small and expand in terms of \( \lambda \) as follows:
\[ A_1 = a_0 + A_2 + A_3 + A_4 + \ldots \]  
(20a)
\[ A_2 = b_0 + A_2 + A_3 + A_4 + \ldots \]  
(20b)
\[ A_3 = c_0 + A_3 + A_4 + A_5 + \ldots \]  
(20c)

Substituting from (20a) into (19a), we get, for the first three orders in \( A_i \), the following equations:
\[ \cot \theta A_1^{(1)} + A_1^{(1)} - \lambda A_1 = 0 \]  
(21a)
\[ \cot \theta A_2^{(2)} + A_2^{(2)} - 2 \lambda A_2 = 0 \]  
(21b)
\[ \cot \theta A_3^{(3)} + A_3^{(3)} - 3 \lambda A_3 = 0 \]  
(21c)

Multiplying (2.20a) by \( \sin \theta \), the equation becomes
\[ \frac{d}{d\theta} (\sin \theta A_1^{(1)}) - A_1^{(1)} \sin \theta = 0 \]  
(22)

Assuming \( a_0 \) to be constant that is free of \( \theta \), integrating (22) we get,
\[ a_1 = -a_0 \log(\sin \theta) + h_2 \log \left( \tan \frac{\theta}{2} \right) + h_2 \]  
(23)

where \( h_2 \) and \( h_2 \) are constants.

Then substituting (23) into (21b) and multiplying by \( \sec \theta \), then integrating we get,
\[ a_1 = \frac{2}{3} a_0 \log(\tan \frac{\theta}{2}) - \left( \log(\tan \frac{\theta}{2}) \right)^3 \]  
(24)

We are not proceeding further for \( a_1 \), but for \( \theta = 0, \pi, \) and \( a_0, a_2 \) have singularity and \( a_0 \) likely has the same property.
From (19b) and (20a,b) we have,
\[ 2b_x + \tan \theta b'_x + b''_x = 0 \]  
(25a)
\[ 2b_y + \tan \theta b'_y + b''_y = 3aF \cos \theta \]  
(25b)
where \(a_F\) is a constant. The equation (25a) has the solution, \(b_x = \xi \cos \theta\)  
(26)
where \(\xi\) is a constant.  
(25b) then becomes
\[ 2b_x + \tan \theta b'_x + b''_x = 3aF \xi \cos \theta \]  
(27)
To solve this we put, \(b_x = \cos \theta b_1 + b_2\)  
(28)
Substituting in (27) we get
\[ b_2 = -3aF \xi \left[ \log \sin \theta - \sec \theta \log(\tan \theta) + k' \sec \theta \right]. \]  
(29)
where \(k'\) is an arbitrary constant.

Then we get,
\[ b_1 = -3aF \xi \left( \cos \theta \log(\sin \theta) - \log(\tan \theta) + k' \right) \]  
(30)
Here also for \(\theta = 0, \pi\) \(b_1\) and \(b_2\) have similar situation.

Considering (19c) and (20a,b,c). we get,
\[ 6C_1 + \cot \theta \xi C'_1 + C''_1 = 0 \]  
(31a)
\[ 6C_2 + \cot \theta \xi C'_2 + C''_2 = 3aF \xi \cos \theta \]  
(31b)
(31a) is a Legendre equation for \(l = 2\), we get \(C_1 = \xi \eta (3 \cos^2 \theta - 1)\),  
(32)
\(\eta\) is a constant.

To solve we get, \(C_2 = C_1 \xi\).  
(33)
So that we have,
\[ \left(1 + 2 \xi C \right) f'' + \xi C'' = p \cos^2 \theta + q \]  
(34)
where \(p = 3m \xi^2 + 3 \xi \eta, \quad q = -2a \eta \).

Multiplying (34) by \(\cosh \theta\) and integrating
\[ c'(\theta) = (k - \xi \cos^2 \theta - q \cos \theta)/(C \sin \theta) \]  
(35)
which can be integrated and is obtained from (32). and can be also determined by similar procedure and we get singularity at \(\theta = 0\). It seems that it is not easy to get solution with smooth behavior in \(\theta\) when there is non-linearity.

### 3. MULTIPLE MOMENTS FOR EINSTEIN-MAXWELL EQUATIONS

For convenience, we may repeat some EM equations. The metric used here is as follows:
\[ ds^2 = f(\delta - \omega d\theta)^2 - \rho^{-2} f' \rho_0^2 \rho d^2 - e^{\mu}(d\rho^2 + d\psi^2) \]  
(36)
Where \(f, \omega, \rho, \mu, \phi, \phi'\) are functions of \(\rho\) and \(\psi\).

In terms of the electromagnetic potential \(\phi, \phi'\), the EM equations consist of the following four equations, for four unknown functions \(f, \omega, \rho, \mu, \phi, \phi'\) (Islam [11], P-69),
\[ \left( \begin{array}{c} f \\ \omega \end{array} \right) \left( \begin{array}{c} f' \\ \omega' \end{array} \right) = \left( \begin{array}{c} f' \\ \omega' \end{array} \right) - \frac{1}{\rho} \left( \begin{array}{c} f \\ \omega \end{array} \right) = \left( \begin{array}{c} f \\ \omega \end{array} \right) - \frac{1}{\rho} \left( \begin{array}{c} f' \\ \omega' \end{array} \right) \]  
(37a)
\[ \left( \begin{array}{c} \phi \\ \phi' \end{array} \right) \left( \begin{array}{c} \phi' \\ \phi'' \end{array} \right) = \left( \begin{array}{c} \phi' \\ \phi'' \end{array} \right) - \frac{1}{\rho} \left( \begin{array}{c} \phi \\ \phi' \end{array} \right) \]  
(37b)
\[ \left( \begin{array}{c} \phi \\ \phi' \end{array} \right) \left( \begin{array}{c} \phi'' \\ \phi''' \end{array} \right) = \left( \begin{array}{c} \phi'' \\ \phi''' \end{array} \right) - \frac{1}{\rho} \left( \begin{array}{c} \phi \\ \phi' \end{array} \right) \]  
(37c)
\[ \left( \begin{array}{c} \phi \\ \phi' \end{array} \right) \left( \begin{array}{c} \phi'' \\ \phi''' \end{array} \right) = \left( \begin{array}{c} \phi'' \\ \phi''' \end{array} \right) - \frac{1}{\rho} \left( \begin{array}{c} \phi \\ \phi' \end{array} \right) \]  
(37d)

The function \(\phi\) is given by quadrature
\[ \mu_\phi = -f^{-1} f' + \frac{1}{2} \rho^2 \rho_0^2 \rho_0^2 = 2 \rho f^{-1}(\rho^2 - \rho_0^2) + 2 \rho f^{-1}(\rho^2 - \rho_0^2) \]  
(38a)
\[ \mu_\phi = -f^{-1} f' + \frac{1}{2} \rho^2 \rho_0^2 \rho_0^2 = 2 \rho f^{-1}(\rho^2 - \rho_0^2) + 2 \rho f^{-1}(\rho^2 - \rho_0^2) \]  
(38b)
where \(f, \omega, \rho, \mu, \phi, \phi'\) are determined.

Replacing \(f\) by \(\theta = f^{-1}\).  
(39)
The equations (37a, b, c, d) are reduced to the forms (Islam [11], P-77),
\[ h(5^2 h') - \frac{5^2}{5} = -2 \rho^2 (\rho^2 + \phi^2 + \phi^2 + \phi^2) \]  
(40a)
\[ h(\Delta r) = 2 h_r h_r + h_r h_r = \frac{\rho h^3 (\rho^2 - \rho^2 \phi^2)}{2 h_0^2} \]  
(40b)
\[ h(2 \phi - \phi - \phi - \phi) = -2 h_0 \phi \phi - \phi - \phi - \phi \]  
(40c)
\( h \nabla^2 \phi' = -h_y \phi' + h_y \phi_{z} + \rho^{-3}(w_y \phi_z - w_z \phi_y) \) \hspace{1cm} (40d) 

where, \( \nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \phi^2} \).

\( \nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \phi^2} \).

We now transform \( (r, \theta) \) to Boyer-Lindquist coordinates \( (r, \theta) \) as follows:

\( \rho = \left( r^2 - 2mr + \omega^2 \right)^{\frac{1}{2}} \sin \theta, \theta = (r - m) \cos \theta \) \hspace{1cm} (41) 

Then the equations (40 a,b,c,d) take the following forms:

\[ h \left( r^2 - 2mr + \omega^2 \right) r_{rr} + 2(\theta - 2mr + \omega^2) \phi_{r} - 2(\theta - 2mr + \omega^2) \phi_{\theta} = -2h \left( r^2 - 2mr + \omega^2 \right) \]

\[ (\phi_{r} + \phi_{\theta}) + \phi_{\phi} \]

\[ h \left( r^2 - 2mr + \omega^2 \right) r_{rr} + 2(\theta - 2mr + \omega^2) \phi_{r} - 2(\theta - 2mr + \omega^2) \phi_{\theta} = -2h \left( r^2 - 2mr + \omega^2 \right) \]

\[ (\phi_{r} + \phi_{\theta}) + \phi_{\phi} \]

\[ \left( r^2 - 2mr + \omega^2 \right) r_{rr} + 2(\theta - 2mr + \omega^2) \phi_{r} - 2(\theta - 2mr + \omega^2) \phi_{\theta} = -2h \left( r^2 - 2mr + \omega^2 \right) \]

\[ (\phi_{r} + \phi_{\theta}) + \phi_{\phi} \]

Now we consider a power series expansion, in inverse powers of \( r \), for the functions \( h, w, \phi, \phi' \) as follows.

\[ h(r, \theta) = h_0(\theta) + h_1(\theta) + h_2(\theta) + h_3(\theta) + \ldots \] \hspace{1cm} (43a) 

\[ w(r, \theta) = w_0(\theta) + w_1(\theta) + w_2(\theta) + w_3(\theta) + \ldots \] \hspace{1cm} (43b) 

\[ \phi(r, \theta) = \phi_0(\theta) + \phi_1(\theta) + \phi_2(\theta) + \phi_3(\theta) + \ldots \] \hspace{1cm} (43c) 

\[ \phi'(r, \theta) = \phi'_0(\theta) + \phi'_1(\theta) + \phi'_2(\theta) + \phi'_3(\theta) + \ldots \] \hspace{1cm} (43d) 

It is well known that for any bounded rotating source, the boundary equations at large distance from the source for the functions \( h(r, \theta) \) and \( w(r, \theta) \) are as follows, as \( r \to \infty; \)

\[ h = 1 + \frac{\Delta}{2r}, \quad w = \frac{\Delta n^2 M}{r} \] \hspace{1cm} (44) 

where \( M \) and - \( \frac{\Delta}{2} \) are respectively the total mean and total angular momentum of the system; This \( M \) is same as in (41).

Thus in (43a,b,c,d) we can set

\[ h_0(\theta) = 1, \quad h_1(\theta) = 2M, \quad w_0(\theta) = 0, \quad w_1(\theta) = 3\Delta n^2 M \] \hspace{1cm} (45) 

We write \( h_{20} = \frac{4\Delta}{20}, \quad w_{20} = \frac{6\Delta}{20} \), etc.

We have form (43a)

\[ h_2 = \frac{4\Delta}{10} - \frac{2\Delta}{10} - \frac{4\Delta}{10} \] \hspace{1cm} (46a) 

\[ h_2 = \frac{4\Delta}{10} + \frac{2\Delta}{10} + \frac{4\Delta}{10} \] \hspace{1cm} (46b) 

\[ h_2 = \frac{2\Delta}{10} + \frac{2\Delta}{10} \] \hspace{1cm} (46c) 

\[ h_2 = \frac{2\Delta}{10} + \frac{2\Delta}{10} \] \hspace{1cm} (46d) 

and similar relations for \( w, \phi, \phi' \) respectively. For (42a), we also have the expansion of the factor as

\[ (r^2 - 2mr + \omega^2)^{-1} = \frac{1}{r^2} + \frac{2m}{r^4} + \frac{2 \Delta n^2 M}{r^4} + \ldots \] \hspace{1cm} (47) 

The resulting forms of equations (42a,b,c,d) after insertion from(46a,b,c) and similar relations for \( w, \phi, \phi' \) are given in Appendix I.

Now we write down below the first three equations from each of the four equations, corresponding to inverse powers of \( r \):

\[ r^{-1} : 2w_{20} + w_{42} - \cot \theta w_{22} = \pm \ln(\phi_1 \phi'_{22} - \phi'_1 \phi_{22}) \] \hspace{1cm} (48a)
We expand $\phi_{z}$ as follows (for convenience omit the subscript in the function on the right hand side).

$$\phi_{z} = \phi_{z}^{(0)} + \lambda \phi_{z}^{(1)} + \lambda^{2} \phi_{z}^{(2)} + \cdots$$

(61a)

$$\phi_{z}^{(0)} = \phi_{z}^{(1)} + \lambda \phi_{z}^{(2)} + \cdots$$

(61b)

Here $\phi_{z}^{(1)}$ and $\phi_{z}^{(0)}$ are function of $\theta$.

Normally, when $\lambda = 0$, that is the angular momentum vanishes the magnetic part of the electromagnetic field also vanishes. However, in (61a,b), assuming that $\phi_{z}^{(0)}$ and $\phi_{z}^{(1)}$ are non-zero function of $\theta$, both the potential do not vanish for $\lambda = 0$. In that case the potentials appearing here should be regarded as combination of electric and magnetic potentials, or perhaps one may attribute a magnetic monopole term to the magnetic field. With this cautionary note we will go ahead to examine the consequence of the expansions (61a,b).

For the first orders, that is, zero, first and second order in $\lambda$, we get the following equations:

$$\phi_{z}^{(0)} + \cot \theta \phi_{z}^{(1)} + 2 \phi_{z}^{(2)} = 0$$

(62a)

$$\phi_{z}^{(0)} + \cot \theta \phi_{z}^{(1)} + 2 \phi_{z}^{(2)} = 0$$

(62b)

$$\phi_{z}^{(0)} + \cot \theta \phi_{z}^{(1)} + 2 \phi_{z}^{(2)} = 4 \cos \theta \phi_{z}^{(0)} - \sin \theta \phi_{z}^{(1)}$$

(62c)

$$\phi_{z}^{(0)} + \cot \theta \phi_{z}^{(1)} + 2 \phi_{z}^{(2)} = \sin \theta \phi_{z}^{(0)} - 4 \cos \theta \phi_{z}^{(1)}$$

(62d)
It is almost a monopole of the source vanish; at.

For (62a,b), where $A_2$ and $A_1$ are arbitrary constants. (62c) becomes,

$$
\phi_{m} = A_2 \cos \theta + B_2
$$

(63)

We seek a solution of the form satisfying (64)

$$
\phi = A_3 \sin^2 \theta + B_3
$$

(64)

Provided $A_3 = \frac{1}{2} A_2$, $B_3 = \frac{1}{2} A_2$

So that a solution of (64) is

$$
\phi = \frac{1}{2} A_2 \sin^2 \theta + \frac{1}{2} A_2 + u \omega \theta
$$

(65)

The last part arises due to the homogeneous part of (64).

Similarly, substituting from (63) for $\phi_{m}$ the right hand side of (62d) we get

$$
\phi_{m} = \frac{1}{6} A_3 \sin^2 \theta = -3A_2 \cos^2 \theta = A_2
$$

(66)

The solution of which, following (64) and (66), is given as follows:

$$
\phi = \frac{1}{2} A_2 \sin^2 \theta - \frac{1}{2} A_2 + u \omega \theta
$$

(67)

where the last part arises from the homogeneous parts of (67).

Substituting from (68) into the right hand sides of (62e) we get the following equation for $\phi_{m}$.

$$
\phi_{m} = \frac{1}{6} A_3 \sin^2 \theta = -3A_2 \cos^2 \theta = A_2
$$

(68)

The solution of which is given by

$$
\phi = \frac{1}{6} A_3 \sin^2 \theta - \frac{1}{6} A_2 \cos \theta + \frac{1}{6} A_4 \cos \theta
$$

(69)

Comparing (69) and (70)

the solution of (71) is given by

$$
\phi = \frac{1}{6} A_3 \sin^2 \theta - \frac{1}{6} A_2 \cos \theta + \frac{1}{6} A_4 \cos \theta
$$

(71)

Thus, (63), (66), (68), (70) and (72) give a closed form of solutions for the six functions $\phi_{m}$, $\phi_{m}^{(2)}$, $\phi_{m}^{(3)}$, $\phi_{m}^{(4)}$, $\phi_{m}^{(5)}$, $\phi_{m}^{(6)}$. One can attempt to solve these in a power series expansion in $\lambda$, as above.

4. PHYSICAL INTERPRETATION OF THE SOLUTION OBTAINED HERE

The leading (monopole) terms remain the same, whether the radial co-ordinate are the Schwarzschild type or the Boyer-Lingquist. $\phi_{m}^{(2)} = \phi_{m}^{(3)} = 0$, means that total charge and total strength of magnetic monopole of the source vanish; this is true for astrophysical bodies, such as neutron stars. The higher order terms then $\phi_{m}^{(4)}$ in the expansion (43a-d) correspond to multipole moments, such as mass quadru pole moment, electric octupole moment, etc.

Because of the approximate nature of the solution and the difficulties of interpretation we are not able to give an adequate physical interpretation, but only to make some tentative additional remarks.

In the nonlinear model of the last section, although for the linear limit of $\lambda = 0$ one can get solutions (0f Laplace’s equation) which are regular in $\theta$, when the non-linear term $\lambda \phi^2$ is introduced one seems to get solutions which have singularities for $\theta = 0, \pi$. It is almost as if the nonlinear distorts the ‘linearity of force’ of the relevant field in such a manner as to make it difficult to get regular (in $\theta$) solutions. A similar phenomenon seems to take place in the electromagnetic field for although in the pure Einstein rotating solutions, Forrester was able to get solutions regular in $\theta$ in the first few orders of $\gamma^{-1}$ for the functions $h_{m}$, when the electromagnetic field is introduced (in the potential $\phi_{m}$), one gets singularities in $\theta$ at $\theta = 0, \pi$. It thus appears that the electromagnetic field distorts the pure gravitational field in such a manner as to make it difficult to get solutions regular in $\theta$.

These appear to be some similarities in the functions that appear in the transition from the linear to the non –linear model on the one hand, and that from the pure gravitational field to a combination of the gravitational and electromagnetic field. Further investigation may reveal more interesting connections.

REFERENCES

APPENDIX-I

\[
\frac{1}{r^2} \left( \frac{1}{r^2} - 2 \ln r + \alpha^2 \right) \left( \frac{1}{r^2} - 2 \ln r + \alpha^2 \right) + 2 \left( \frac{1}{r^2} - \frac{1}{r^2} - \frac{1}{r^2} - \frac{1}{r^2} - \cdots \right) + \frac{1}{r^2} \left( \frac{1}{r^2} - 2 \ln r + \alpha^2 \right) \left( \frac{1}{r^2} - 2 \ln r + \alpha^2 \right) + 2 \left( \frac{1}{r^2} - \frac{1}{r^2} - \frac{1}{r^2} - \frac{1}{r^2} - \cdots \right)
\]

\[
= -2 \left( \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \cdots \right) \left( \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \cdots \right)
\]

\[
(\text{A1})
\]

\[
\left( \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \cdots \right) \left( \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \cdots \right) + 2 \left( \frac{1}{r^2} - \frac{1}{r^2} - \frac{1}{r^2} - \frac{1}{r^2} - \cdots \right) + \frac{1}{r^2} \left( \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \cdots \right) + 2 \left( \frac{1}{r^2} - \frac{1}{r^2} - \frac{1}{r^2} - \frac{1}{r^2} - \cdots \right)
\]

\[
= -2 \left( \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \cdots \right) \left( \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} + \cdots \right)
\]

\[
(\text{A2})
\]
APPENDIX-II

In this appendix we write down the four third order equations obtained from (A₃), … (A₄) by setting equal to zero the coefficients of $r^{-3}$ in these equations. These may be of some use in any future consideration of the problem.

\[
6b_2 - 12m_2h_2 + 4ma^2 + h_{3pp} + \cot 0h_20 + 2m(h_{2pp} + \cot 0h_20) = 0 \quad (A'_1)
\]

\[
12m_2 - 8m_2n_2 - 2h_2n_2 + 2w^2_1n_2 + w_2 - \cot \theta n_2 + 2m(w_{2pp} - \cot \theta n_2) + h_2(w_{2pp} - \cot \theta n_2) - 2h_20 w_{20} = 8 \sin \theta (\phi_{21} \phi'_{21} - \phi_{21}^2 \phi_{21}) \quad (A'_2)
\]

\[
6h_2 + \cot \theta \phi_{20} + 2m(h_{2pp} + \cot \theta \phi_{20}) = \frac{1}{\sin \theta} \left( 2(\phi_{21} w_{20} - w_{21} \phi_{20}) + \left(3\phi_{21} w_{20} - w_{21} \phi_{20} + 2m(w_{21} \phi_{20} - 2\phi_{21} w_{20}) \right) \right) \quad (A'_3)
\]

\[
6\phi_{21} + \phi'_{2pp} + \cot \theta \phi'_{21} + 2m(\phi'_{2pp} + \cot \theta \phi'_{21}) = \frac{1}{\sin \theta} \left( 2(\phi_{21} w_{20} - w_{21} \phi_{20}) + \left(3\phi_{21} w_{20} - w_{21} \phi_{20} - 2m(w_{21} \phi_{20} - 2\phi_{21} w_{20}) \right) - 2m(\phi_{21} w_{20} - w_{21} \phi_{20}) \right) \quad (A'_4)
\]

Note that electromagnetic potentials do not occur in the equation for $r^{-2}$. As before, the equations being somewhat intractable, one can consider power series expansion in $\lambda$, which would correspond to slow rotation, in which the functions found earlier are to be used for the expansion of $\phi_{21}, \phi'_{21}$, etc. It is clear that singularities at $\theta = 0, \pi$ are likely to persist. Higher order equations in $r^{-2}$ have a similar structure.

BIOGRAPHY

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