Unsteady Hydromagnetic Natural Convection Couette Flow through a Vertical Channel in the Presence of Thermal Radiation under an Exponentially Decaying Pressure Gradient with Viscous and Joule Dissipation Effects Using Galerkin’s Finite Element Method

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Abstract
This paper examines the unsteady hydromagnetic free convection Couette flow of an incompressible Newtonian fluid in the presence of thermal radiation and an exponentially decaying pressure gradient. The applied uniform magnetic field is perpendicular to the plates and is assumed to be fixed relative to the moving plate. The momentum equation considers buoyancy forces, while the energy equation considers the effects of thermal radiation and viscous and Joule dissipations. The fluid is considered to be a gray absorbing-emitting but non-scattering medium in the optically thick limit. The radiative heat flux is described by the Rosseland approximation in the energy equation. The Galerkin finite element method is used to discretize the coupled pair of non-linear partial differential equations. The resulting system of non-linear algebraic equations is solved using an iterative method to obtain the velocity and temperature distributions. The effects of radiation parameter $R_a$, Grashof number $Gr$, magnetic parameter $H$, Prandtl number $Pr$ and Eckert number $E$, on both the velocity and temperature distributions are investigated.

Keywords: MHD, Natural Convection, Thermal Radiation, Couette Flow, Exponentially Decaying Pressure Gradient, Finite Element Method

1. INTRODUCTION
Magnetohydrodynamics (MHD) is an area that has been studied extensively by many researchers owing to its many industrial applications such as the use of MHD pumps in chemical energy technology [10], the operations of MHD accelerators and the purification of crude oil [4]. The steady MHD flow between two infinite parallel stationary plates in the presence of a transverse uniform magnetic field was first studied in 1937 by Hartmann and Lazarus ([1], [19]). Rossow [19] considered the hydromagnetic flow over a flat plate in the presence of a transverse magnetic field in the cases where the magnetic field is fixed relative to the plate and fixed relative to the fluid. It was found that when the magnetic field is considered to be fixed relative to the plate, the flow is accelerated by the magnetic field. In the case where the magnetic field is fixed relative to the fluid, the magnetic field retards the fluid flow. Attia and Abdeen [4] investigated the unsteady Hartmann flow between two infinite parallel plates with viscous and Joule dissipation under an exponentially decaying pressure gradient. An analytical solution for the velocity distribution was obtained using the Laplace transform technique and the temperature distribution was obtained using the finite difference method. The results showed that the velocity decreases as the magnetic field strength increases. Increasing the strength of the magnetic field was shown to increase the temperature at small times but decrease the temperature at large times. It was also observed that increasing the suction parameter results in a decrease in both the velocity and temperature.

Couette flow is important flow phenomenon with respect to engineering applications involving shear-driven flow such as aerodynamic heating and polymer technology [6]. In particular, the study of MHD Couette flow is useful for acquiring a better understanding of electrostatic precipitation and MHD power generators [3]. Katagiri (1962) extended the work of Hartmann and Lazarus to the case of unsteady Couette flow [20]. The Laplace transform technique was used to obtain exact solutions for the velocity field and skin friction on the surface of the lower plate. Muhuri (1963) further extended this study by considering the effects of uniform suction and injection between porous walls when one of the walls is uniformly accelerated [20]. Attia and Ewis [3] considered the problem of unsteady MHD Couette flow of a viscoelastic fluid under an exponentially decaying pressure gradient with suction and injection through the plates and viscous and Joule dissipations and obtained graphical representations of the velocity and temperature distributions using the finite
The graphical results show that the magnetic field strength and suction parameter affect the velocity and temperature in the same way as observed in [4].

The study of free convection flow is important in understanding natural circulation in geothermal reservoirs, problems involving the spread of pollution and other energy-related engineering applications [17]. Chamkha [5] studied the steady hydromagnetic mixed convection flow in a vertical channel with symmetric and asymmetric wall heating conditions in the presence or absence of heat generation or absorption. In the case where the effects of Joule and viscous dissipations are included, the problem was solved using the finite difference method. It was found that the Joule and viscous dissipations increased the fluid velocity and temperature.

Singh [20] investigated the effect of natural convection on unsteady Couette flow. The Laplace transform technique was used to obtain the velocity and temperature fields, the skin-friction and rate of heat transfer. It was observed that an increase in the Grashof number results in an increase in the flow velocity. The work of Singh [20] was extended by Jha [10] by discussing the combined effects of natural convection and a uniform transverse magnetic field when the magnetic field is fixed relative to the plate or fluid. Using the Laplace transform technique, exact solutions were obtained for the velocity and temperature fields. The trends observed with respect to the magnetic parameter were consistent with those observed in [19].

Thermal radiation effects in fluid flow are significant in many industrial applications where high temperatures are involved. In engine combustion chambers, furnaces and power plants for gas cooled nuclear reactors; thermal radiation has a considerable effect on heat transfer [15]. In the limit of an optically thick medium, the radiative flux is related to the temperature using the Rosseland Approximation [14]. Rajput and Sahu [16] considered the unsteady natural convection hydromagnetic Couette flow between two infinite vertical plates in the presence of thermal radiation. The magnetic field was assumed to be fixed relative to the moving plate and the velocity and temperature distributions were obtained in the cases of impulsive and uniformly accelerated movement of the plate. The authors discovered that the velocity and temperature decrease with increasing Prandtl number and with increasing radiation parameter in both cases. It was also observed that increasing the magnetic parameter increases the velocity in the case of impulsive movement of the plate but decreases the velocity in the case of uniformly accelerated movement of the plate. Increasing the Grashof number results in an increase in the velocity in both cases. Job and Gunakala [11] extended this study by considering the effects of viscous and Joule dissipations. Approximate solutions for the velocity and temperature distributions were obtained using the finite element method. It was observed the fluid velocity and temperature increase with increasing magnetic parameter, Grashof number and Eckert number. An increase in the radiation parameter and Prandtl number increases the velocity and temperature of the fluid at short time but increases them at large time.

This paper extends the work of Job and Gunakala [11] by assuming the presence of an exponentially decaying pressure gradient. The pair of coupled non-linear partial differential equations which governs this problem is solved using Galerkin’s finite element method.

### 2. MATHEMATICAL ANALYSIS

Consider the incompressible flow of a Newtonian fluid between two parallel vertical non-conducting plates which are located on planes $y = 0$ and $y = h$, and are infinite in the $x$ and $z$ directions. The plate at $y = h$ is stationary and the other plate moves with time-dependent velocity $U_0 e^{ct}$ in the positive $x$-direction (where $U_0$ is constant and $c$ is a non-negative integer). The temperature of the moving and stationary plates are fixed at $T_1$ and $T_2$ respectively, with $T_1 > T_2$. An exponentially decaying pressure gradient is applied at $t = 0$ in the positive $x$-direction. A magnetic field with magnitude $B_0$, which is fixed relative to the moving plate, is applied in the positive $y$-direction. We make the following simplifying assumptions.

1. The magnetic Reynolds number is very small.
2. For a typical conductor, $\rho_0$ is very small; hence it is negligible.
3. The Boussinesq approximation is applied.
4. The fluid is a gray and optically thick absorbing-emitting but non-scattering medium.
5. The fluid has a refractive index of unity.

Under the above assumptions, the governing equations are:

### Continuity equation:
Momentum equation:
\[ \rho \frac{\partial u}{\partial t} = \frac{\partial p}{\partial x} + \rho g \alpha (T - T_2) - \gamma p_n(u - U_0) + \mu \frac{\partial^2 u}{\partial y^2} \]  

Energy equation:
\[ \rho c_p \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial y^2} - \frac{\partial q_r}{\partial y} + \mu \left( \frac{\partial u}{\partial y} \right)^2 + \gamma p_n u^2 \]  

where \( u \) is the flow velocity in the \( x \)-direction, \( \rho \) is the fluid density, \( p \) is the fluid pressure, \( g \) is the acceleration due to gravity, \( \alpha \) is the coefficient of thermal expansion, \( T \) is the fluid temperature, \( \gamma \) is the electrical conductivity, \( \mu \) is the viscosity, \( c_p \) is the specific heat capacity at constant pressure, \( \kappa \) is the thermal conductivity and \( q_r \) is the radiative heat flux.

The boundary conditions are:
\[ u(y,0) = 0, u(h,t) = 0 \]  
\[ T(y,0) = T_2, T(0,t) = T_1, T(h,t) = T_2 \]  

The radiative flux is simplified using the Rosseland approximation to give
\[ q_r = -\frac{4\sigma T^4}{3b_R} \]  

where \( \sigma \) is the Stefan-Boltzmann constant and \( b_R \) is the mean absorption coefficient.

Using this approximation in equation (6) gives
\[ q_r = -\frac{16\sigma T^2}{3b_R} \frac{\partial T}{\partial y} \]  

which on substituting into equation (3) gives
\[ \rho c_p \frac{\partial T}{\partial t} = \left( \kappa + \frac{16\sigma T^2}{3b_R} \right) \frac{\partial^2 T}{\partial y^2} + \mu \left( \frac{\partial u}{\partial y} \right)^2 + \gamma p_n u^2 \]  

The following dimensionless quantities are defined:
\[ \tilde{x} = \frac{x}{h}, \tilde{y} = \frac{y}{h}, \tilde{u} = \frac{u}{U_0}, \tilde{h} = \frac{h}{U_0}, \tilde{T} = \frac{T - T_2}{T_1 - T_2}, \tilde{Gr} = \frac{\rho \beta h^2 \alpha (T_1 - T_2)}{\mu U_0}, \]  
\[ H = \frac{\gamma p_n U_0}{\mu}, a = \frac{\rho h^3}{\mu}, Pr = \frac{\kappa}{\gamma p_n}, R_d = \frac{\kappa b_R}{4\sigma T_2^4}, E_c = \frac{U_0^2}{c_p(T_1 - T_2)} \]  

On dropping all hats and taking the dimensionless pressure gradient to be \( -\frac{\partial p}{\partial x} = Ce^{-\beta t} \) for constants \( C \) and \( \beta \), equations (2) and (7) become
\[ \frac{\partial u}{\partial t} = Ce^{-\beta t} + Gr T - \tilde{H}(u - a \tilde{r}) + \frac{\partial^2 u}{\partial y^2} \]  
\[ \frac{\partial T}{\partial t} = \left( \frac{3R_d + 4}{3R_d Pr} \right) \frac{\partial^2 T}{\partial y^2} + E_c \left( \frac{\partial u}{\partial y} \right)^2 + HE_c \tilde{u}^2 \]  

with boundary conditions
\[ u(y,0) = 0, u(0,t) = a \tilde{r}^c, u(1,t) = 0 \]  
\[ T(y,0) = 0, T(0,t) = 1, T(1,t) = 0 \]

where \( Gr, H, a, Pr, R_d \) and \( E_c \) are the Grashof number, magnetic parameter, accelerating parameter, Prandtl number, radiation parameter and Eckert number respectively.

3. Numerical Solution by the Finite Element Method

3.1 Spatial Semi-discretization

The spatial domain (0,1) is discretized into elements \( \Omega^k = (y_k^l, y_k^r) \) for \( k = 1, 2, \ldots, r \). For each \( \Omega^k \), the Lagrange quadratic interpolation functions \( \psi^k_i, \psi^k_2 \) and \( \psi^k_3 \) are used as basis functions for the approximations \( u_k \) and \( T_k \) of \( u \) and \( T \) respectively. Under the Galerkin finite element method, we choose the weight functions to be the \( \psi^k_i \), for \( i = 1, 2, 3 \). According to the weighted integral formulation [18], (9) and (10) become:
3.2 Time Discretization

We discretize the time domain \((0,L)\) into elements \(\Omega^k = (t_n, t_{n+1})\) for \(n = 1, 2, \ldots, N - 1\), where the Lagrange linear interpolation functions \(\phi_1\) and \(\phi_2\) are chosen to be basis functions for \(\xi^k_j\) and \(\eta^k_j\). Under the Galerkin finite element method, we choose the weight functions to be the \(\phi_s\), for \(s = 1, 2\). The weighted integral form for (16) and (17) is given by:

\[
\left( \sum_{j=1}^{n} A_{ij} \frac{d \xi_j}{dt} + \sum_{j=1}^{n} B_{ij} \frac{d \eta_j}{dt} - Q^k_i - J^k_i \right) \phi_i \, dt = 0
\]

\[
\left( \sum_{j=1}^{n} A_{ij} \frac{d \eta_j}{dt} + \sum_{j=1}^{n} B_{ij} \frac{d \xi_j}{dt} + \sum_{j=1}^{n} R^k_i \right) \phi_i \, dt = 0
\]

We assume the following forms:

\[
\xi^k_j(t) = \sum_{m=1}^{n+1} \xi^k_m \phi_{m-1}(t), \eta^k_j(t) = \sum_{m=1}^{n+1} \eta^k_m \phi_{m-1}(t), f^k_i(t) = \sum_{m=1}^{n+1} f^k_m \phi_{m-1}(t),
\]

\[
Q^k_i(t) = \sum_{m=1}^{n+1} Q^k_m \phi_{m-1}(t), R^k_i(t) = \sum_{m=1}^{n+1} R^k_m \phi_{m-1}(t), f^k_i = f^k_i(t_m), Q^k_i = Q^k_i(t_m), R^k_i = R^k_i(t_m), p^{k}_{ij} = p^{k}_{ij}(z^k_p(t_m)).
\]

On rearranging, substituting (20) into (18) and (19) and taking \(s = 2\), we get the fully discretized element equations in matrix form:

\[
\begin{bmatrix}
3A^k + 2A^k \left( B_{ij}^{k1} + 3B_{ij}^{k3} \right)
2A^k B_{ij}^{k2}
6A^k + 4A^k B_{ij}^{k4}
\end{bmatrix}
\begin{bmatrix}
U^k_n
X^k_n
\end{bmatrix}
+
\begin{bmatrix}
F^k_n + Q^k_n + 2(Q^k_{n+1})
F^k_n
\end{bmatrix}
\]

where \(A^k = (A_{ij}^k)\) is the local mass matrix, \(B_{ij}^{k1} = (B_{ij}^{k1})\), \(B_{ij}^{k2} = (B_{ij}^{k2})\), \(B_{ij}^{k3} = (B_{ij}^{k3})\) and \(B_{ij}^{k4} = (B_{ij}^{k4})\) are the local stiffness matrices, \(U^k_n = (U^k_n)\) is the local vector of position coefficients, \(X^k_n = (X^k_n)\) is the local vector of velocity coefficients, and \(F^k_n + Q^k_n = (f^k_n + Q^k_n)\) and \(R^k_n = (R^k_n)\) are the local force vectors.
The spatial domain is divided into 40 line elements, while the time domain is divided into 500 line elements. After assembly of the elements and applying the given boundary conditions (11) and (12), a system of 81 non-linear equations is obtained at each time level. An iterative method is used to determine an approximate solution for this system of equations to an accuracy of $5 \times 10^{-4}$.

4. RESULTS AND DISCUSSION

The finite element solution for the velocity and temperature was computed for different values of the radiation parameter, Grashof number, magnetic parameter, Prandtl number and Eckert number in the cases of impulsive ($c=0$) and uniformly accelerated ($c=1$) movement of the plate at $y=0$. The numerical results were analysed by considering the following values of the above parameters: radiation parameter $R_d = 0.1, 1, 10$; Grashof number $Gr = 1, 5, 10$; magnetic parameter $H = 2, 4, 6$; Prandtl number $Pr = 0.71$ (for air), 3 (for the saturated liquid Freon at 273.3K), 7 (for water) and Eckert number $Ec = 0.2, 1, 2$. Throughout the analysis, the parameters $C$ and $\beta$ are taken to be 5 and 1.2 respectively.

The velocity and temperature profiles for the current finite element solution with parameter $C$ set to zero was compared with the previous results of Job and Gunakala [11] in figures 2 and 3. It was observed that the present numerical results are in excellent agreement with that obtained in [11].

![Figure 2: Comparison of Velocity Profiles in the Case of Uniformly Accelerated Movement of the Plate](image1)

![Figure 3: Comparison of Temperature Profiles in the Case of Uniformly Accelerated Movement of the Plate](image2)

The effect of the Eckert number $Ec$ on the velocity and temperature of the fluid at the centre of the channel ($y=0.5$) is shown in figures 4-7. The Eckert number is proportional to thermal energy dissipation throughout the fluid. The velocity and temperature was observed to increase with increasing $Ec$. The temperature increase observed in figures 6 and 7 may be attributed to an increase in the viscous and Joule dissipations. The associated increase in velocity occurs via free convection currents (as shown in figures 4 and 5).

![Figure 4: Time Development of Velocity for Different Values of $Ec$ in the Case of Impulsive Movement of the Plate](image3)
Figure 5: Time Development of Velocity for Different Values of $E_c$ in the Case of Uniformly Accelerated Movement of the Plate

Figure 6: Time Development of Temperature for Different Values of $E_c$ in the Case of Impulsive Movement of the Plate

Figure 7: Time Development of Temperature for Different Values of $E_c$ in the Case of Uniformly Accelerated Movement of the Plate

Figures 8-11 show the effect of the Grashof number $Gr$ on the velocity and temperature of the fluid at the centre of the channel ($y=0.5$). The Grashof number is proportional to the buoyancy forces associated with free convection within the fluid. Figures 8 and 9 show that an increase in $Gr$ causes an increase in the flow velocity. It is observed from figures 10 and 11 that the temperature also increases with increasing $Gr$. This occurs as a result of the viscous and Joule dissipations which are associated with the observed increased velocity.

Figure 8: Time Development of Velocity for Different Values of $Gr$ in the Case of Impulsive Movement of the Plate
Figure 9: Time Development of Velocity for Different Values of Gr in the Case of Uniformly Accelerated Movement of the Plate

Figure 10: Time Development of Temperature for Different Values of Gr in the Case of Impulsive Movement of the Plate

Figure 11: Time Development of Temperature for Different Values of Gr in the Case of Uniformly Accelerated Movement of the Plate

The effect of the magnetic parameter H on the velocity and temperature at the centre of the channel (y=0.5) is displayed in figures 12-15. From figure 12, the velocity is shown to increase with increasing H in the case of impulsive movement of the plate. Increasing the magnetic parameter causes an increase in the magnetic field strength and the associated Lorentz force. The fluid is pulled more strongly by the magnetic field (which is considered to be fixed relative to the moving plate) and consequently the flow velocity increases. Figure 13 reveals that in the case of uniformly accelerated movement of the plate, an increase in the magnetic parameter increases the flow velocity at small time but increases it at large time. Near the initial time t = 0 the velocity of the moving plate is low, while the magnitude of the pressure gradient is high. Consequently, the velocity of the fluid at the centre of the channel is greater than the velocity of the plate (as shown in figure 16). Therefore, at small time the action of the Lorentz force is to decelerate the fluid. However, as time progresses, the pressure gradient decays while the Lorentz force on the fluid increases. Hence, the flow velocity becomes dominated by the uniformly accelerated movement of the plate over time (see figure 16). Since the magnetic field is fixed relative to the uniformly accelerated plate, the flow is accelerated by the magnetic field at large time. From figures 14 and 15 it is seen that increasing H also increases the fluid temperature. This temperature increase is caused by Joule dissipation within the fluid, which is proportional to H.
Figure 12: Time Development of Velocity for Different Values of H in the Case of Impulsive Movement of the Plate

Figure 13: Time Development of Velocity for Different Values of H in the Case of Uniformly Accelerated Movement of the Plate

Figure 14: Time Development of Temperature for Different Values of H in the Case of Impulsive Movement of the Plate

Figure 15: Time Development of Temperature for Different Values of H in the Case of Uniformly Accelerated Movement of the Plate
Figures 17-20 show the effects of the radiation parameter $R_d$ on the fluid velocity and temperature at the centre of the channel ($y=0.5$). The results show that increasing $R_d$ causes a decrease in the velocity and temperature at small time but increases them at large time. Viscous and Joule dissipations are negligible at small time since the flow velocity is low near $t = 0$. Furthermore, an increase in the amount of radiation emitted by the fluid, which is represented by $R_d$, reduces the rate of heat transfer through the fluid. This accounts for the observed decrease in temperature with increasing $R_d$ at small time. This temperature decrease causes a reduction in buoyancy forces within the fluid, thereby decreasing the flow velocity. However, viscous and Joule dissipations have a significant impact on temperature at large times. Therefore, when the velocity becomes sufficiently high over time, the temperature increases as $R_d$ is increased and through natural convection effects the velocity of the fluid also increases.
Figures 21-24 display the effects of the Prandtl number Pr on velocity and temperature at the centre of the channel (y=0.5). Increasing Pr is observed to decrease the velocity and temperature at for a short time but increase them over longer time periods. Since the velocity is small near the initial time, viscous and Joule dissipations are negligible. If the Prandtl number is increased through a decrease in thermal conductivity, the temperature of the fluid decreases at small times, which also decreases the flow velocity as a result of reduced buoyancy forces. Since the Prandtl number is also proportional to the fluid viscosity, the observed decrease in fluid velocity with increased Prandtl number is consistent with the expectation that a fluid with a higher viscosity has a lower velocity. On the contrary, viscous and Joule dissipation effects are significant at large time since the flow velocity increases over time. In this case, the fluid temperature increases with increasing Pr, which results in an increase in velocity due to the enhanced buoyancy forces.
4. CONCLUSION

In this paper, the unsteady MHD free convection Couette flow of an incompressible viscous fluid in the presence of thermal radiation and an exponentially decaying pressure gradient was investigated. The effects of radiation parameter $R_d$, Grashof number $Gr$, magnetic parameter $H$, Prandtl number $Pr$ and Eckert number $E_c$ on both the velocity and temperature distributions have been investigated. The radiation parameter, Eckert number and Prandtl number were discovered to have a greater effect on the temperature than on the velocity. However, the velocity is affected to a greater extent by the Grashof number and magnetic parameter. It was found that the radiation parameter and Prandtl number decrease the fluid velocity and temperature at small time but increase them at large time. The magnetic parameter is shown to be proportional to the fluid temperature. In the case of impulsive movement of the plate, the velocity increases with increasing magnetic number. However, increasing the magnetic parameter causes the flow velocity to decrease at small time and increase at large time in the case of uniformly accelerated movement of the plate. Lastly, increasing the Grashof number and Eckert number increase the fluid velocity and temperature in both cases.

REFERENCES


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